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A Class of One-Step Embedded Numerical Integrator for the Solution of Stiff Ordinary Differential Equations

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Abstract

Due to severe step size restriction, it becomes absolutely necessary that only methods with large regions of absolute stability remain suitable for stiff equations. In this paper, the continuous Block one-step embedded numerical integrator of order 4 and order 5 have been constructed. The continuous scheme was evaluated at different points to obtain discrete schemes. The order, error constant, zero stability and consistency of the resulting discrete schemes were ascertained. The region of absolute stability of the block hybrid scheme was plotted. Their accuracy and stability investigated shows that the new methods are A(α)- stable, a property desirable of numerical methods suitable for the solutions of stiff ODE's. The new one-step embedded Numerical Integrators used in block form tested on stiff systems of Ordinary Differential Equations confirms that they are efficient and they compare favorably with exact solution and the state of the art Matlab ODE solver, ode15s. (ode 15s is for stiff problems while ode 45s are for non-stiff problems)

1.0 Introduction

The limitation of analytical means in finding an exact solution to most modeled equations cannot be over emphasized. It then becomes necessary to apply numerical methods when faced with such problems. By numerical method we mean a difference equation involving the number of consecutive approximation $y_{n+j}, j = 0, 1, ... k$, from which it will be possible to compute sequentially the sequence $y_n,\ n = 0, 1, \ldots N$; the k is called the step number of the method. These numerical methods use the available discrete numerical integration algorithms in which the numerical approximations are obtained at some specific points in the interval of integration.

Initial value problems (IVP) is defined as any differential equation of the form

$$\left.\begin{aligned} y'(x) &= f(x\, y(x)), \\ y(x) &= y_0, a \le x \le b \end{aligned}\right\} \tag{1.1}$$

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on a given mesh,

$a = x_0 < x_2 < ...... < N = b$ With a mesh size N

The numerical solution of (1.1) is a major focus of this paper.

The linear multistep method (LMM), though efficient with regards to accuracy for a given number of functions evaluation per step, suffer the pitfall of poor stability property as step number increases. However, it was observed that some of the difficulties inherent in the linear multistep -methods can be reduced by lowering the step number and increasing the order without reducing the stability interval. This gives rise to the idea of a hybrid scheme. It is called hybrid because it possess some properties of LMM and that of Runge-Kutta methods (Stells & Gragg in Butcher & Burrage, 2004).

We therefore define K-hybrid scheme as follows:

$$\sum \alpha j \quad y_{n+j} = h \quad \sum_{j=0}^{k} \sum_{j=0}^{k} \beta \text{j} \quad f_{n+j} + h\beta_v f_{n+v}$$

where $\alpha k = 1$, α_0 and B_0 are not both zero and V € {0, 1… k} and $f_{n+v} = f(k_{n+v}, y_{n+v})$ which is the off-grid function evaluation. One main disadvantage of the hybrid scheme is that they require special predictor to predict the off-grid points. In this paper, this problem is solved with the use of block method. The block method is purposed to ease computational effort and prevent the use of starting values for the hybrid discrete schemes. These schemes are obtained by evaluating the various continuous schemes at both grid and off-grid points as the case may be, which are then simplified to obtain the block method of each scheme.

2.0 Derivation Techniques

Consider the initial value problem (1.1)

$y' = f(x, y); a \leq x \leq b, y(a) = yo$,

on a given mesh $a = xo < x_1 < x_2 < ... < x_m = b$, where $h = x_{n+1} - x_n, n = 0,1,n,$ where h is a constant step and k is the step number of the method.

In order to solve equation (1.1) Onumanyi etal (1994) developed a linear multistep method with continuous coefficient by the idea of multistep collocations.

Let $y(x) = \sum_{j=0}^{t-1} \alpha j(x) y(x_{n+i}) + h \sum \beta(x) f(\bar{x} j, \bar{y}(xj))$ (2.1)

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where $\alpha j(x) = \sum_{i=0}^{k+m-1} \alpha j,\ i+1\, x^i$

$$h\beta j(x) = \sum_{i=0}^{k+m-1} h\beta j, i+1x^i$$

Also $\begin{array}{l} y(x_{n+j}) = jn + j,\ \ j \in (0,1,\ldots k=1) \\ y1(\bar{x}j) = f(xj, y(\bar{x}j)_{,}|,\ \ j = 1,\ldots\ldots m \end{array}$

to get $\alpha j(x)$ and $\beta_j^{(x)}$, Sirisena [2003] used the matrix equation of the form

$$DC = I \tag{2.2}$$

where I is the identity matrix of dimension (t+m) x (t+m) while D and C are matrices defined as:

$$D=\begin{pmatrix} 1, x_n, x_n^{\ 2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_n^{\ t+m-1} \\ 1, x_{n+1}, x^2{}_{n+1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, x_{n+1}^{\ t+m-1} \\ ::\qquad\qquad\qquad\qquad : \\ ::\qquad\qquad\qquad\qquad : \\ ::\qquad\qquad\qquad\qquad : \\ 1, x_{n+t-1}, x^2{}_{n+t-1}, \ldots\ldots\ldots\ldots\ldots.., x_{n+t-1}^{\ t+m-1} \\ 0,\quad 1,\quad 2x_0 \ldots\ldots\ldots\ldots.., (t+m\text{-}1)x_0^{\ t+m-2} \\ :\qquad\qquad\qquad : \\ :\qquad\qquad\qquad : \\ :\qquad\qquad\qquad : \\ 0,\quad 1,\quad 2x_{m-1} \ldots\ldots\ldots\ldots, x_{m-1}^{\ t+m\text{-}2} \end{pmatrix} \tag{2.3}$$

The above matrix (2.3) is the multistep collocation matrix of dimension $(t+m)\times(t+m)$ and

$$C=\begin{pmatrix} \alpha_{0.1}, & \alpha_{11}\ --- & \alpha t\ \text{-}_{1,1}, & h\beta_{01} & h\beta_{m\text{-}1,1} \\ \alpha_{02} & \alpha_{1.2}\ --- & \alpha t\ \text{-}_{1.2}, & h\beta_{02} & h\beta_{m\text{-}1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1_{\alpha 0,\ t+m} & \alpha_{1,t+m} & \alpha_{t\text{-}1,\ t+m} & h\beta_{0t+m} & h\beta_{m\text{-}1,t+m} \end{pmatrix} \tag{2.4}$$

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We define t as the number of interpolation points used while m is the number of collocation points used. The columns of the matrix C =D[-1] gives the continuous coefficients $\alpha_j(x),\ j=0,1,k-1$ and $\beta_j(x),\ j=0,1,.....k-1$

3.0 Block Hybrid Explicit Methods of Step Number K=4

Consider the discrete Adams Bashforth method of order 2

$$y_{n+2}=y_{n+3}+\frac{h}{24}\left[55f_{n+3}-59f_{n+2}+37f_{n+1}-9f_n\right] \qquad (3.1)$$

Expressing (3.1) in the general form gives

$$y(x)=\alpha_1^{(x)}y_{n+}+h\left[\beta o^{(x)}f_n+\beta_1^{(x)}f_{n+1}+\beta_1^{(x)}f_{n+\frac{3}{2}}+\beta_1^{(x)}f_{n+2}+\beta_1^{(x)}f_{n+2}+\beta_1^{(x)}f_{n+3}\right] \qquad (3.2)$$

Introducing one off grid collocation points at $x=x_{n+1/4}, x=x_{n+1/2}$, $x=x_{n+\frac{3}{4}}, x=x_{n+5/4}$ *and* $x=x_{n+1}$, In x gives

$$y(x)=\beta_0(x)hf_n+\beta_1(x)hf_{n+1}+\left[\alpha_1 y_n+\alpha_{\frac{1}{4}}(x)y_{n+\frac{1}{4}}+\alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}}+\alpha_{\frac{3}{4}}(x)y_{n+\frac{3}{4}}+\alpha_{\frac{5}{4}}(x)y_{n+\frac{5}{4}}+\alpha_1(x)y_{n+1}\right]$$

(3.3)

and the resultant matrix is given below as,

$$D=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & \left(x_n+\frac{1}{3}h\right) & \left(x_n+\frac{1}{3}h\right)^2 & \left(x_n+\frac{1}{3}h\right)^3 & \left(x_n+\frac{1}{3}h\right)^4 \\ 1 & \left(x_n+\frac{2}{3}h\right) & \left(x_n+\frac{2}{3}h\right)^2 & \left(x_n+\frac{2}{3}h\right)^3 & \left(x_n+\frac{2}{3}h\right)^4 \\ 1 & \left(x_n+\frac{2}{3}h\right) & \left(x_n+\frac{2}{3}h\right)^2 & \left(x_n+\frac{2}{3}h\right)^3 & \left(x_n+\frac{2}{3}h\right)^4 \\ 1 & (x_n+h) & (x_n+h)^2 & (x_n+h)^3 & (x_n+h)^4 \\ 0 & 1 & (2x_n+2h) & 3\,(x_n+h)^2 & 4(x_n+h)^3 \end{pmatrix}$$

(3.4)

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The elements of matrix C are obtained from the inverse of D by the use of maple 7 software and after simplification we obtain the continuous form expressed as

$$y(x) = U_1.y_n + U_2.y_{n+\frac{1}{3}} + U_3.y_{n+\frac{2}{3}} + U_4.y_{n+1} + U_5.f_{n+1} \tag{3.5}$$

Evaluating (3.5) at $x = x_n,\ x_n +_{1/3},\ x_n +_{2/3}\ and\ x_{n+1}$ yields the following discrete schemes as shown in 4.0 below.

4.0 One step Embedded Method of Order 4

$$y_{n+1} = -\left(\frac{6}{197}\right)hf_{n+\frac{4}{3}} + \left(\frac{50}{197}\right)hf_{n+1} + \left(\frac{279}{197}\right)y_{n+\frac{2}{3}} - \left(\frac{99}{197}\right)y_{n+\frac{1}{3}} + \left(\frac{17}{197}\right)y_n$$

$$y_{n+\frac{1}{3}} = -\left(\frac{1}{3}\right)hf_{n+\frac{1}{3}} + \left(\frac{1}{9}\right)hf_{n+1} + 2y_{n+\frac{2}{3}} - \left(\frac{7}{9}\right)y_{n+1} - \left(\frac{2}{9}\right)y_n$$

$$y_{n+\frac{2}{3}} = -\left(\frac{2}{3}\right)hf_{n+\frac{2}{3}} - \left(\frac{2}{9}\right)hf_{n+1} - y_{n+\frac{1}{3}} + \left(\frac{17}{9}\right)y_{n+1} + \left(\frac{1}{9}\right)y_n$$

$$y_{n+\frac{4}{3}} = \left(\frac{4}{3}\right)hf_{n+1} - 2y_{n+\frac{1}{3}} + 6y_{n+\frac{2}{3}} - \left(\frac{10}{3}\right)y_{n+1} + \left(\frac{1}{3}\right)y_n \tag{4.1}$$

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & \left(x_n + \frac{1}{4}h\right) & \left(x_n + \frac{1}{4}h\right)^2 & \left(x_n + \frac{1}{4}h\right)^3 & \left(x_n + \frac{1}{4}h\right)^4 & \left(x_n + \frac{1}{4}h\right)^5 \\ 1 & \left(x_n + \frac{1}{2}h\right) & \left(x_n + \frac{1}{2}h\right)^2 & \left(x_n + \frac{1}{2}h\right)^3 & \left(x_n + \frac{1}{2}h\right)^4 & \left(x_n + \frac{1}{2}h\right)^5 \\ 1 & \left(x_n + \frac{3}{4}h\right) & \left(x_n + \frac{3}{4}h\right)^2 & \left(x_n + \frac{3}{4}h\right)^3 & \left(x_n + \frac{3}{4}h\right)^4 & \left(x_n + \frac{3}{4}h\right)^5 \\ 1 & (x_n + h) & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 \\ 0 & 1 & (2x_n + 2h) & 3\,(x_n + h)^2 & 4(x_n + h)^3 & 5(x_n + h)^4 \end{pmatrix}$$

(3.4)

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The elements of matrix C are obtained from the inverse of D by the use of maple 7 software and after simplification we obtain the continuous form expressed as

$$y(x) = U_1.y_n + U_2.y_{n+\frac{1}{4}} + U_3.y_{n+\frac{1}{2}} + U_4.y_{n+\frac{3}{4}} + U_5.y_{n+1} + U_6.f_{n+1} \tag{3.5}$$

Evaluating (3.5) at $x = x_n, x_n +_{3/4}, x_{n+1}, \ x_{n+2} x = x_n, x_n +_{3/2}, x_{n+31}$ and x_{n+4} yields the following discrete schemes as shown in 4.0 below.

4.1 Onestep Embedded Method of Order 5

$$y_{n+1} = -\left(\frac{36}{2501}\right)hf_{n+\frac{5}{4}} + \left(\frac{411}{2501}\right)hf_{n+1} + \left(\frac{4008}{2501}\right)y_{n+\frac{3}{4}} + \left(\frac{728}{2501}\right)y_{n+\frac{1}{4}} - \left(\frac{2124}{2501}\right)y_{n+\frac{1}{2}} - \left(\frac{111}{2501}\right)y_n$$

$$y_{n+\frac{1}{4}} = -\left(\frac{3}{14}\right)hf_{n+\frac{1}{4}} - \left(\frac{3}{56}\right)hf_{n+1} - \left(\frac{9}{7}\right)y_{n+\frac{3}{4}} + \left(\frac{27}{14}\right)y_{n+\frac{1}{2}} + \left(\frac{29}{56}\right)y_{n+1} - \left(\frac{9}{56}\right)y_n$$

$$y_{n+\frac{1}{2}} = -\left(\frac{1}{2}\right)hf_{n+\frac{1}{2}} + \left(\frac{1}{12}\right)hf_{n+1} + \left(\frac{8}{3}\right)y_{n+\frac{3}{4}} - \left(\frac{8}{9}\right)y_{n+\frac{1}{4}} - \left(\frac{31}{36}\right)y_{n+1} + \left(\frac{1}{12}\right)y_n$$

$$y_{n+\frac{3}{4}} = -\left(\frac{3}{2}\right)hf_{n+\frac{3}{4}} - \left(\frac{3}{8}\right)hf_{n+1} + y_{n+\frac{1}{4}} - \left(\frac{9}{2}\right)y_{n+\frac{1}{2}} + \left(\frac{37}{8}\right)y_{n+1} - \left(\frac{1}{8}\right)y_n$$

$$y_{n+\frac{5}{4}} = \left(\frac{5}{4}\right)hf_{n+1} + \left(\frac{5}{3}\right)y_{n+\frac{1}{4}} - 5y_{n+\frac{1}{2}} + 10y_{n+\frac{3}{4}} - \left(\frac{65}{12}\right)y_{n+1} - \left(\frac{1}{4}\right)y_n \tag{4.1}$$

The scheme is consistent and zero stable, hence it is convergent.

In summary, the block method has the following order and error constants

Table 1 **Order and Error Constants of order 4**

Evaluating Point $X = x_n$	Order 4	Error constants $\frac{37}{159570}$
$X = x_{n+\frac{1}{3}}$	4	$\frac{-1}{7290}$
$X = x_{n+\frac{2}{3}}$	4	$\frac{1}{7290}$
$X = x_{n+1}$	4	$\frac{1}{1215}$

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Table 2 **Order and Error Constants of order 5**

Evaluating Point	Order 5	Error constants
$X = x_n$		$\frac{197}{25610240}$
$X = x_{n+\frac{1}{4}}$	5	$\frac{3}{573440}$
$X = x_{n+\frac{1}{2}}$	5	$\frac{-1}{184320}$
$X = x_{n+\frac{3}{4}}$	5	$\frac{1}{81920}$
) $X = x_{n+\frac{5}{4}}$	5	$\frac{1}{24576}$

The methods are all convergent since there are all consistent and zero-stable.

5.0 Stability Regions of the Block Hybrid Explicit Methods

To plot the region of absolute stability of the block hybrid method, the newly constructed methods are reformulated as General Linear Methods and expressed as

$$\begin{bmatrix} y \\ y_{i+1} \end{bmatrix} = \left\langle \frac{A}{B} \middle| \frac{U}{V} \right\rangle \left[\begin{pmatrix} hf(y) \\ y_{n+1} \end{pmatrix} \right] \tag{5.1}$$

where **A**= $\begin{bmatrix} a_1 & K & a_{1s} \\ a_s & K & a_{ss} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & K & b_{1s} \\ b_S & K & B_{SS} \end{bmatrix}$

The elements of u and v are obtained from the interpolation and collocation points respectively. The element of the matrices A, B, U and V are substituted into the stability matrix

$$m(z) = B_2 + ZA_2(I - ZA_1)^{-1} B_1 \tag{5.2}$$

and the stability function $P(\eta, z) = \det(\eta I - M(z))$ (5.3)

$$Y=\begin{bmatrix} y_n \\ y_{n+1} \\ M \\ y_{n+k} \end{bmatrix}, \quad y_{i+1}=\begin{bmatrix} y_{n+k} \\ y_{n+k-1} \end{bmatrix}, \quad y_{i-1}=\begin{bmatrix} y_{n+k-1} \\ y_{n+k-2} \end{bmatrix} \tag{5.4}$$

Using maple 7 software gives the stability polynomial below

$$f(z)=\frac{144}{197}r^4-\frac{144}{197}r^3-\frac{120}{197}r^4z+\frac{140}{591}r^4z^2-\frac{100}{1773}r^4z^3+\frac{16}{1773}r^4z^4-\frac{16}{1773}r^3z^3-\frac{72}{197}r^3z$$
$$-\frac{44}{591}r^3z^2-\frac{120}{197}r^4+\frac{280}{591}r^4z-\frac{100}{591}r^4z^2+\frac{64}{1773}r^4z^3-\frac{4}{197}r^3z^2-\frac{72}{197}r^3-\frac{88}{591}r^3z$$
(5.5)

Using MATLAB software and stability polynomial of ONIM of order 4,

$$f(z)=-\frac{38880}{17507}r^5z+\frac{13770}{17507}r^5z^2-\frac{6075}{35014}r^5z^3+\frac{3699}{140056}r^5z^4-\frac{405}{140056}r^5z^5+\frac{51840}{17507}r^5-\frac{51840}{17507}r^4$$
$$-\frac{25920}{17507}r^4z-\frac{810}{2501}r^4z^2-\frac{675}{17507}r^4z^3-\frac{81}{35014}r^4z^4-\frac{38880}{17507}r^5+\frac{27540}{17507}r^5z-\frac{18225}{35014}r^5z^2$$
$$+\frac{3699}{35014}r^5z^3-\frac{2025}{140056}r^5z^4-\frac{25920}{17507}r^4-\frac{1620}{2501}r^4z-\frac{2025}{17507}r^4z^2-\frac{162}{17507}r^4z^3$$
(5.6)

Using MATLAB software and stability polynomial of ONIM of order 5, the stability region of the block methods are plotted and are shown to be A(α) -SATBLE (See fig 5.1)

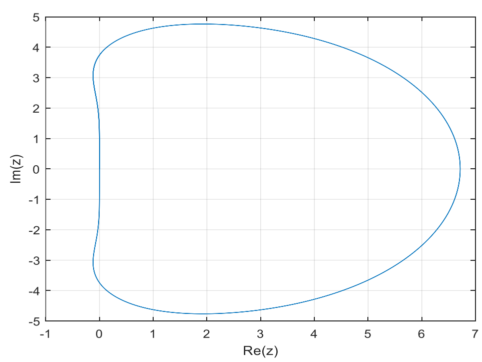

Fig 1 Region of Absolute Stability of Onim Of Order 4

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From the graph the ONIM of order 4 is A(α) -stable

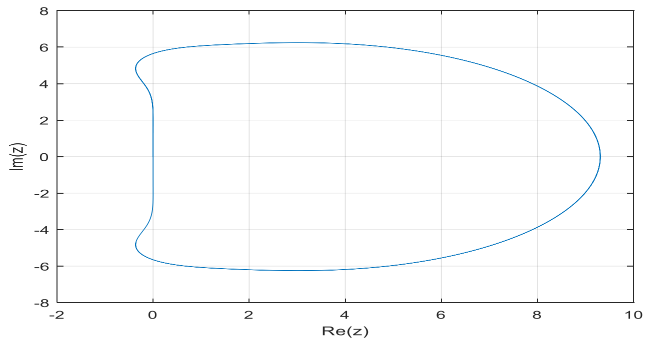

Fig 2 Region of Absolute Stability of ONIM of Order 5

From the graph the ONIM of order 5 is A(α) -stable

6. Numerical Experiment

We report here a numerical example on stiff problems taken from the literature using the solution curve and numerical result. In comparison, we also report the performance of the new one-step enbedded block multistep methods and the well-known Matlab stiff ODE solver ODE15S on the same problems and on the same axes.

Problem 1: Consider the system

$$y_1' = -8y_1 + 7y_2$$
$$y_2' = 42y_1 - 43y_2$$
$$y_1(x) = 2e^{-x} - e^{-50x}, y_2(x) = 2e^{-x} + 6e^{-50x}$$
$$y_1(0) = 1, y_2(0) = 8, \ 0 \le x \le 1$$

Table 6. Numerical results for Problem 1.

h	Method	MAXE
10^{-2}	HBDDF(5)	2.37429×10^{-1}
	BBDF(5)	1.97128×10^{-1}
	OUR METHOD(order 4)	7.1041×10^{-4}
	OUR METHOD(order 5)	1.96108×10^{-5}
	ode15s	5.02584×10^{-2}
10^{-4}	HBDDF(5)	9.49700×10^{-5}

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BBDF(5)	3.94490×10^{-4}
OUR METHOD(order 4)	1.10285×10^{-8}
OUR METHOD(order 5)	1.85252×10^{-8}
ode15s	9.96987×10^{-4}

HBDDF(5) found in the paper below

1. Ibrahim, ZB & Nasarudin, AA 2020, 'A class of hybrid multistep block methods with a-stability for the numerical solution of stiff ordinary differential equations', Mathematics, vol. 8, no. 6, 914.https://doi.org/10.3390/math8060914
2. BBDF(5) found in
   Nasir, N.A.A.M.; Ibrahim, Z.B.; Suleiman, M.B.; Othman, K.I. Fifth Order Two-point Block Backward
   Differentiation Formulas for Solving Ordinary Differential Equations. Appl. Math. Sci. 2011, 5, 3505–3518.
   Problem 2: Consider the system

$$y_1' = -y_2 - \frac{y_1 y_3}{r}$$

$$y_2' = y_1 - \frac{y_2 y_3}{r} \qquad \text{where } r = \sqrt{y_1^2 + y_2^2}$$

$$y_3' = \frac{y_1}{r}$$

$$y_1(x) = (2+\cos x)\cos x, y_2(x) = (2+\cos x)\sin x, y_3(x) = \sin x$$

$$y_1(0) = 3, y_2(0) = 0, y_3(0) = 0,\ 0 \le x \le 20, h = 0.01$$

The result here is compared with the exact solution and the ode15s

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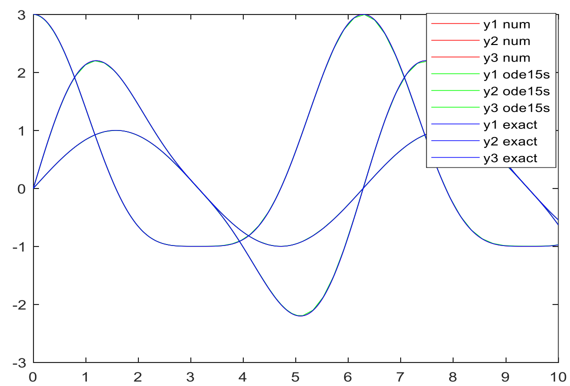

Fig 3: Solution using one step order 4 method

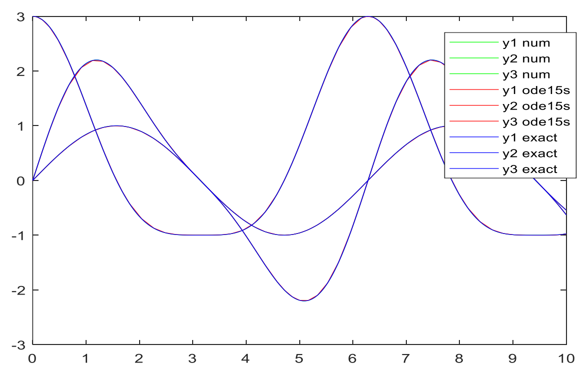

Fig 4 : Solution curve using one step of order 5

Problem 3: Consider the linear mildly stiff system

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The exact solution to the system is given as

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 4e^{-x} - 3e^{-1000x} \\ -2e^{-x} + 3e^{-1000x} \end{pmatrix}$$

The stiffness ratio of the system is 1:1000.

x	y_i	Error in 3SHM10 of order 5 in [X]	Error in 3SHM20 of order 6 in [X]	Error in our method of order 4	Error in our method of order 5

5	y_1	1.2352×10^{-13}	2.1372×10^{-15}	1.8830×10^{-8}	9.5356×10^{-11}
	y_2	2.3527×10^{-13}	8.7325×10^{-16}	9.4150×10^{-9}	4.7678×10^{-11}
40	y_1	3.1562×10^{-16}	2.3372×10^{-18}	8.7696×10^{-23}	4.8099×10^{-25}
	y_2	3.1625×10^{-16}	4.1783×10^{-18}	4.3848×10^{-23}	2.4049×10^{-25}
70	y_1	4.2561×10^{-19}	3.289×10^{-21}	1.4361×10^{-35}	7.8764×10^{-38}
	y_2	3.6172×10^{-20}	3.2891×10^{-22}	7.1805×10^{-36}	3.9382×10^{-38}

[X] Joshua S,Chibuisi C, Ezekiel O. O. and John B. G. A Pair of Three-Step Hybrid Block Methods for the Solutionsof Linear and Nonlinear First-Order Systems, *European Journal of Mathematics and Statistics*,2022; 3(1):1-25.

Observation: Our methods of order 4 and 5 perform better than the methods of order 5 and 6 which are of higher orders than ours which implies that our compete favourably with existing solutions.

The solutions curves in figure 2 shows that our method BHEM compete favourably with the ODE solver 15s. From problem 2 tables 2 shows that our methods performed well with marginal absolute error constants. From the stiff ode problems solved, BHEM tend to converge much faster to the theoretical solution. The methods is therefore recommended for the solutions of mildly initial value problem.

Conclusion

Our new One-step embedded Block methods have been constructed through the multistep collocation approach. The Region of absolute stability of the ONIM has been greatly enhanced. (A(α) -stable). These methods are all convergent. Numerical results reveal the efficiency of the methods in solving stiff systems equations.

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