



A Fuzzy Inventory Model for Deteriorating Items in a Pharmaceutical Industry with Shortages and Partial Backlogging

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Abstract

A fuzzy inventory model for deteriorating items was proposed. The rate of deterioration was assumed to be time dependent and followed a three-parameter Weibull distribution. The demand rate is considered to be price dependent, shortages are allowed, and are partially backlogged. A crisp model was first developed, and thereafter, the ordering, holding, deterioration, shortage, and lost sale costs and other parameters were assumed to be triangular fuzzy numbers. The graded Mean Integration Representation, Signed Distance and Centroid methods were applied to defuzzify the total average cost function, and the results were compared.

Keywords: Fuzzy, Partial Backlogging

1. Introduction

An organization's daily operations heavily rely on inventory management, which involves keeping track of stock levels and making timely orders to restock items. Effective management of inventory helps reduce costs and ensure the hitch-free process of an organization (Poswal et al. 2022). Essentially, inventory can be categorized into finished goods, work-in-progress, and raw materials inventory. The primary aim of maintaining inventory is to maintain a balance between the minimization of the total cost and maximization of customer satisfaction. It is important to consider the effect of deterioration when modeling the inventory of certain items, such as fruits, vegetables, and chemicals, whose value reduces with time. Deterioration can be defined as the spoilage, damage, evaporation, expiration, decay, etc. of items over time. Some key parameters, such as the deterioration rate and demand function in inventory models, can be assumed to be deterministic or probabilistic in nature. Many researchers have assumed constant deterioration and demand rate functions. However, in reality, these functions are unpredictable and cannot be estimated accurately. Hence, fuzzy concepts must be applied to estimate them when formulating inventory models. To develop a more realistic inventory model, quantities such as deterioration, demand rates, and all costs related to inventory models are considered as fuzzy numbers. This is because fuzziness is a close possible approach to reality. The concept of fuzzy set theory was first introduced by Zadeh (1965), which provides a framework for considering

parameters that are not clearly defined, vague, or imprecise values assigned or determined based on the subjective judgements of individuals. Subsequently, Zadeh and Bellman (1970) considered an inventory model for decision making in a fuzzy environment. Kao and Hsu (2002), developed a single period inventory model with fuzzy demand. Syed and Aziz (2007) considered the signed distance method for a fuzzy inventory model without any shortages. Jaggi et al. (2012) developed a fuzzy inventory model for deteriorating items with time varying demand and shortages. Dutta and Kumar (2012) developed a fuzzy inventory model without shortages, using trapezoidal fuzzy numbers. Dutta and Kumar (2013) considered an optimal replenishment policy for an inventory model without shortages by assuming fuzziness in the demand, holding cost, and ordering cost. Whitin (1957) first considered the effect of deterioration in a study related to the management of fashion goods. Kumar and Rajput (2015) developed an inventory model for deteriorating items with time dependent demand rate and partial backlogging. The demand, deterioration, and backlogging rates were assumed to be triangular fuzzy numbers. Shabani et al. (2015) developed a two-warehouse inventory model with fuzzy deterioration and demand rates under a conditionally permissible delay-in-payments. The maxi-mini principle was used in their model to minimize the fuzzy total cost function. Mandal and Islam (2015) considered a constant demand function and deterioration that followed a Weibull distribution to develop a fuzzy two-warehouse inventory model. Shortages were allowed and fully backlogged, and cost components, such as holding, deterioration, and shortage costs, were considered as triangular fuzzy numbers. They applied a graded mean integration representation for defuzzification of the total cost function. Sayal et al., (2018a) studied an inventory model in crisp and fuzzy environment without shortage using triangular fuzzy number. Sayal et al. (2018b) developed crisp and fuzzy inventory models for perishable items. The demand function in that model was considered a ramp-type function, and shortages were allowed. Kumar (2021) proposes a fuzzy inventory model with shortages. In the model, the holding and ordering costs were considered as continuous functions of time. An inventory model with demand and deteriorating rate that follows a two-parameter Weibull distribution was studied by Sayal et al. (2022). Murthy, (2017) proposed an inventory control model with Weibull deterioration rate and logarithmic demand rate for the optimal stock of commodities to meet the future demand. Yadav et al. (2017) studied a two-warehouse inventory fuzzy model for non-instantaneous deteriorating items with conditionally permissible delay-in-payments. The cost components are assumed to be triangular fuzzy numbers. The signed distance method was used to minimize the total cost function. Singha et al. (2018) developed a fuzzy inventory model for deteriorating items with stock-dependent demand rates. The graded mean integration representation, signed distance, and centroid methods were applied for defuzzification. A fuzzy inventory with Weibull deterioration rate, quadratic demand rate, and variable holding cost under permissible shortages was developed by Poswal et al. (2022a). In their model, costs such as ordering, shortage, and deteriorating costs were considered pentagonal fuzzy numbers and defuzzified using a graded mean representation. Poswal et al. (2022b) studied an economic ordering policy model to control deteriorating medicinal products of uncertain demand with trade credit for healthcare industries. The signed distance technique is employed to defuzzify the total cost of the system.

2. Definitions and Preliminaries

Definition 2.1: (Kumar & Rajput, 2015)

Let X be a nonempty set. Then a fuzzy set $\bar{A}$ on X is defined by

$\bar{A} = \{(x, \lambda_{\bar{A}}(x)) : x \in X\}$ where $\lambda_{\bar{A}} : X \rightarrow [0,1]$ is called the membership function and $\lambda_{\bar{A}}(x)$ is the degree of x in $\bar{A}$.

Definition 2.2: (Kumar & Rajput, 2015)

A fuzzy number $\bar{A}$ is a fuzzy set on the real line $\mathbb{R}$, if its membership function $\lambda_{\bar{A}}(x)$ has the following properties.

- (i) $\lambda_{\bar{A}}(x)$ is upper semi continuous
- (ii) $\lambda_{\bar{A}}(x) = 0$, outside some interval $[a_1, a_4]$ there exists real numbers a_2 and a_3 ,
 $a_1 \leq a_2 \leq a_3 \leq a_4$ such that $\lambda_{\bar{A}}(x)$ is increasing on $[a_1, a_2]$, decreasing on $[a_3, a_4]$ and
 $\lambda_{\bar{A}}(x) = 1$, for each x in $[a_2, a_3]$.

Definition 2.3: (Kumar & Rajput, 2015)

A triangular fuzzy number is specified by the triplet (a_1, a_2, a_3) where $a_1 < a_2 < a_3$ and defined by its continuous membership as

$$\lambda_{\bar{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & \text{when } a_1 < x < a_2 \\ \frac{a_3 - x}{a_3 - a_2} & \text{when } a_2 < x < a_3 \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.4: (Jaggi et al., 2012)

If $\bar{A} = (a, b, c)$ is a triangular fuzzy number then the graded mean representation of $\bar{A}$ is defined as
 $G(\bar{A}) = \frac{1}{6}(a + 4b + c)$.

Definition 2.5: (Kumar & Rajput, 2015)

If $\bar{A} = (a, b, c)$ is a triangular fuzzy number then the signed distance of $\bar{A}$ is defined as $d(\bar{A}) = \frac{1}{4}(a + 2b + c)$.

Definition 2.6: (Kumar & Rajput, 2015)

If $\bar{A} = (a, b, c)$ is a triangular fuzzy number then the centroid method on $\bar{A}$ is defined as $C(\bar{A}) = \frac{1}{3}(a + b + c)$.

3 Assumptions of the Study

The development of the new model is based on the following assumptions given below:

1. The demand function depends on selling price of the product and is given by

$$D(p) = a - bp > 0.$$
2. A three parameter Weibull distribution is assumed as product rate of deterioration given as $\alpha\beta(t - \gamma)^{\beta-1}$ where $0 < \alpha < 1$ is the scale parameter, $\beta > 0$ is the shape parameter, $t > 0$ is the time of deterioration and $\gamma \leq t$ is the location parameter.
3. Shortages are allowed and partially backlogged and the rate of backlogging is assumed to be a function which depends on waiting time.
4. Deteriorated items are not being replaced.
5. Learning effect is being applied to both ordering and holding costs.
6. It is assumed that the time taken between placing and receiving an order (lead time) is zero.

Notations

The following notation was adopted for the development of this model:

T	Time per cycle
t_1	Time for which zero inventory level is attained
α, β and γ	Coefficients of deterioration
a, b	Parameters for demand
n	Cycle number
c_1	Per unit purchasing cost
p	Per unit selling price, $p > c_1$
Q	Ordered quantity
Q_1	Inventory initial level at $t = 0$
Q_2	Amount of shortage
O_0	Part of ordering cost that is fixed per order
O_1	Variable part of ordering cost per order that can be reduced owing to learning effect
h_0	Part of holding cost that is fixed per unit
h_1	Variable part of holding cost per unit which can be reduced owing to learning effect
d	Cost of deterioration per unit
s	Cost of shortage per unit
l	Cost of lost sale per unit
u	Coefficient of backlogging, $u > 1$
λ	Positive coefficient for backlogging
Δ_1, Δ_2	Positive constants
$T.A.C.$	Total average cost

3.1 Mathematical Formulation of the Crisp Model

Figure 1 shows the behavior of the system over time. When $t = 0$, the retailer receives items and the inventory level becomes $I_1(0) = Q_1$. Thereafter, the level of inventory starts decreasing owing to the deterioration of items and market demand during the period $[0, t_1]$ until it reaches zero at $t = t_1$. During $[t_1, T]$, shortages are allowed to exist and the demand that occurs within the period is backlogged partially or lost, and the rate of backlogging is assumed to be a function that depends on waiting time. The differential equations describing the behavior of pharmaceutical inventory at any time t are as follows:

$$\frac{dI_1(t)}{dt} = -(a - bp) - \alpha\beta(t - \gamma)^{\beta-1}I_1(t), \quad 0 \leq t \leq t_1 \quad (1)$$

$$\frac{dI_2(t)}{dt} = -(a - bp), \quad t_1 \leq t \leq T \quad (2)$$

$$\text{With boundary conditions } I_1(t_1) = I_2(t_1) = 0 \quad (3)$$

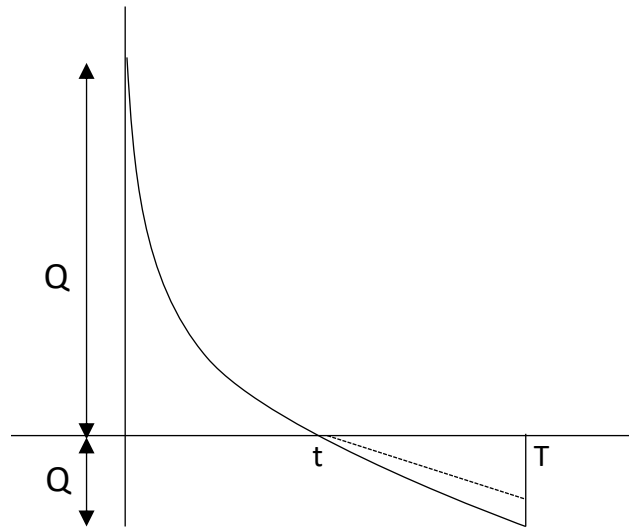


Fig. 1 Inventory Time Graph

Solving (1), we have

$$I_1(t) = (a - bp) \left\{ (t_1 - t) + \frac{\alpha}{\beta+1} [(t_1 - \gamma)^{\beta+1} - (t - \gamma)^{\beta+1}] - \alpha(t_1 - t)(t - \gamma)^{\beta} - \frac{\alpha^2}{\beta+1} (t_1 - \gamma)^{\beta+1}(t - \gamma)^{\beta} + \frac{\alpha^2}{\beta+1} (t - \gamma)^{2\beta+1} \right\}, \quad 0 \leq t \leq t_1 \quad (4)$$

Solving (2) gives

$$I_2(t_1) = (a - bp)(t_1 - t), \quad t_1 \leq t \leq T \quad (5)$$

During a replenishment cycle, the total quantity Q is

$$Q = Q_1 + Q_2 \quad (6)$$

Using equation (4), the initial order quantity Q_1 is given by letting $t = 0$

$$\Rightarrow Q_1 = I_1(0)$$

$$Q_1 = (a - bp) \left\{ t_1 + \frac{\alpha}{\beta+1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^\beta + \frac{\alpha^2 \gamma^\beta}{\beta+1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] \right\} \quad (7)$$

Now the back ordered quantity

$$Q_2 = \int_{t_1}^T u^{-\lambda t} (a - bp) dt$$

$$\Rightarrow Q_2 = (a - bp) \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \quad (8)$$

Using equations (7) and (8) in (6), the total ordering quantity Q is given by

$$Q = (a - bp) \left\{ t_1 + \frac{\alpha}{\beta+1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^\beta + \frac{\alpha^2 \gamma^\beta}{\beta+1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] + \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \right\} \quad (9)$$

Purchasing cost:

The purchasing quantity is given by equation (9), therefore, the purchasing cost is

$$P.C. = c_1 (a - bp) \left\{ t_1 + \frac{\alpha}{\beta+1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^\beta + \frac{\alpha^2 \gamma^\beta}{\beta+1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] + \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \right\} \quad (10)$$

Holding cost:

The inventory cost during the interval $[0, t_1]$ is

$$\begin{aligned}
 H.C. &= \left(h_0 + \frac{h_1}{n^{\Delta_2}} \right) \left\{ \int_0^{t_1} I_1(t) dt \right\} \\
 &= \left(h_0 + \frac{h_1}{n^{\Delta_2}} \right) (a - bp) \left\{ \begin{aligned} &\frac{t_1^2}{2} + \frac{\alpha}{(\beta+1)} \left[t_1(t_1 - \gamma)^{\beta+1} - \frac{(t_1 - \gamma)^{\beta+2}}{(\beta+2)} \right] - \frac{\alpha(t_1 - \gamma)^{\beta+2}}{(\beta+1)(\beta+2)} - \\ &\frac{\alpha^2}{(\beta+1)} \left[\frac{(t_1 - \gamma)^{2\beta+2}}{(\beta+1)} - \frac{(t_1 - \gamma)^{2\beta+2}}{(2\beta+2)} - \frac{2\alpha\gamma^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{\alpha\gamma^{\beta+2}t_1}{(\beta+1)(\beta+2)} \right] \\ &+ \frac{\alpha^2}{(\beta+1)} \left[\frac{\gamma^{2\beta+2}}{(2\beta+2)} - \frac{(t_1 - \gamma)^{\beta+1}}{(\beta+1)} \gamma^{\beta+1} \right] \end{aligned} \right\}
 \end{aligned}
 \tag{11}$$

Deterioration cost:

The total demand during $[0, t_1]$ is

$$\int_0^{t_1} (a - bp) dt$$

Therefore, the number of deteriorating units = $I_1(0)$ – total demand

$$= I_1(0) - \int_0^{t_1} (a - bp) dt$$

Hence, the deterioration cost is given by

$$\begin{aligned}
 &(a - bp) \left\{ t_1 + \frac{\alpha}{\beta+1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^{\beta} + \frac{\alpha^2 \gamma^{\beta}}{\beta+1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] \right. \\
 &\quad \left. + \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \right\} - (a - bp) t_1 \\
 &\Rightarrow D.C. \\
 &= d(a - bp) \left\{ \frac{\alpha}{\beta+1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^{\beta} + \frac{\alpha^2 \gamma^{\beta}}{\beta+1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] \right. \\
 &\quad \left. + \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \right\}
 \end{aligned}
 \tag{12}$$

Shortage cost:

The shortage cost during the interval $[t_1, T]$ is

$$\begin{aligned}
 S.C. &= s \int_{t_1}^T (a - bp) dt \\
 &= s(a - bp)(T - t_1)
 \end{aligned}
 \tag{13}$$

Lost sale cost:

The lost sale cost during the interval $[t_1, T]$ is

$$\begin{aligned} L.S.C. &= l \int_{t_1}^T (1 - u^{-\lambda t})(a - bp) dt \\ &= l(a - bp) \left\{ (T - t_1) + \frac{1}{\lambda \log(u)} (u^{-\lambda T} - u^{-\lambda t_1}) \right\} \end{aligned} \quad (14)$$

Ordering cost:

$$\text{Ordering cost } O.C. = O_0 + \frac{O_1}{n^{\Delta_1}} \quad (15)$$

The total average cost T. A. C. of the system is given by

$$\begin{aligned} T.A.C. &= \frac{1}{T} \left\{ O.C. + P.C. + H.C. + D.C. + S.C. + L.S.C. \right\} \\ &= \frac{1}{T} \left\{ \begin{aligned} &c_1(a - bp) \left\{ t_1 + \frac{\alpha}{\beta + 1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^{\beta} + \frac{\alpha^2 \gamma^{\beta}}{\beta + 1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] \right. \\ &\quad \left. + \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \right\} \\ &+ \left(h_0 + \frac{h_1}{n^{\Delta_2}} \right) (a - bp) \left\{ \begin{aligned} &\frac{t_1^2}{2} + \frac{\alpha}{(\beta + 1)} \left[t_1 (t_1 - \gamma)^{\beta+1} - \frac{(t_1 - \gamma)^{\beta+2}}{(\beta + 2)} \right] - \frac{\alpha (t_1 - \gamma)^{\beta+2}}{(\beta + 1)(\beta + 2)} - \\ &\frac{\alpha^2}{(\beta + 1)} \left[\frac{(t_1 - \gamma)^{2\beta+2}}{(\beta + 1)} - \frac{(t_1 - \gamma)^{2\beta+2}}{(2\beta + 2)} - \frac{2\alpha \gamma^{\beta+2}}{(\beta + 1)(\beta + 2)} \right. \\ &\quad \left. - \frac{\alpha \gamma^{\beta+2} t_1}{(\beta + 1)(\beta + 2)} \right] + \frac{\alpha^2}{(\beta + 1)} \left[\frac{\gamma^{2\beta+2}}{(2\beta + 2)} - \frac{(t_1 - \gamma)^{\beta+1}}{(\beta + 1)} \gamma^{\beta+1} \right] \end{aligned} \right\} \\ &+ d(a - bp) \left\{ \begin{aligned} &\frac{\alpha}{\beta + 1} [(t_1 - \gamma)^{\beta+1} - (-\gamma)^{\beta+1}] - \alpha t_1 (-\gamma)^{\beta} + \frac{\alpha^2 \gamma^{\beta}}{\beta + 1} [(t_1 - \gamma)^{\beta+1} - \gamma^{\beta+1}] \\ &\quad + \left(\frac{u^{-\lambda t_1} - u^{-\lambda T}}{\lambda \log(u)} \right) \end{aligned} \right\} \\ &s(a - bp)(T - t_1) + l(a - bp) \left\{ (T - t_1) + \frac{1}{\lambda \log(u)} (u^{-\lambda T} - u^{-\lambda t_1}) \right\} + \left(O_0 + \frac{O_1}{n^{\Delta_1}} \right) \end{aligned} \right\} \end{aligned} \quad (16)$$

The necessary condition for minimizing T.A.C. is $\frac{\partial T.A.C.}{\partial t_1} = 0$ and $\frac{\partial T.A.C.}{\partial T} = 0$

Now, $\frac{\partial T.A.C.}{\partial t_1}$

$$= \frac{1}{T} \left\{ \begin{aligned} & c_1(a - bp) \left\{ 1 + \alpha[(t_1 - \gamma)^\beta] + \alpha\gamma^\beta + \alpha^2\gamma^\beta[(t_1 - \gamma)^\beta] + u^{-\lambda t_1} \right\} + \\ & \left(h_0 + \frac{h_1}{n^{\Delta_2}} \right) (a - bp) \left\{ \left[t_1 + \alpha(t_1 - \gamma)^{\beta+1} \left(t_1 - \frac{(t_1 - \gamma)}{(\beta + 1)} \right) + (t_1 - \gamma)^{\beta+1} \left(1 - \frac{1}{\beta + 2} \right) \right] - \right. \\ & \quad \left. \alpha t_1(t_1 - \gamma)^{\beta+1} - \frac{\alpha(\beta + 1)}{2} t_1^2(t_1 - \gamma)^\beta + 2(\beta + 1)\alpha^2(t_1 - \gamma)^{2\beta+1} \right\} \\ & \quad - 2t_1 + \gamma \\ & + d(a - bp) \left\{ \alpha(t_1 - \gamma) + \alpha\gamma^\beta + \alpha^2\gamma^\beta(t_1 - \gamma)^\beta \right\} - s(a - bp) \\ & \quad + l(a - bp) \{ (T - t_1) + (u^{-\lambda t_1} - 1) \} \end{aligned} \right\} \\ = 0 \tag{17}$$

Solving $\frac{\partial T.A.C.}{\partial t_1} = 0$, we obtain the optimal value of t_1 which minimize T.A.C. provided it satisfy

$$\frac{\partial^2 T.A.C.}{\partial t_1^2} > 0.$$

$$\frac{\partial^2 T.A.C.}{\partial t_1^2} \\ = \frac{1}{T} \left\{ \begin{aligned} & c_1(a - bp) \left\{ \alpha\beta[(t_1 - \gamma)^{\beta-1}] + \alpha^2\beta\gamma^\beta[(t_1 - \gamma)^{\beta-1}] + u^{-\lambda t_1} \lambda \log(u) \right\} \\ & \left(h_0 + \frac{h_1}{n^{\Delta_2}} \right) (a - bp) \left\{ 1 + \alpha\beta(t_1 - \gamma)^{\beta-1} \left(t_1 - \frac{(t_1 - \gamma)}{(\beta + 2)} \right) + \alpha(t_1 - \gamma)^\beta \left(1 - \frac{1}{\beta + 2} \right) \right\} \\ & \quad + (\beta + 1)(t_1 - \gamma)^\beta \left(1 + \frac{1}{\beta + 2} \right) - \alpha(t_1 - \gamma)^\beta \left(1 - \frac{1}{\beta + 2} \right) \\ & \quad + (\beta + 1)(t_1 - \gamma)^\beta \left(1 + \frac{1}{\beta + 2} \right) - \alpha(t_1 - \gamma)^{\beta+1} - 2\alpha(\beta + 1)t_1(t_1 - \gamma)^\beta \\ & \quad - \frac{\alpha\beta(\beta + 1)}{2} t_1^2(t_1 - \gamma)^{\beta-1} + 2(\beta + 1)(2\beta + 1)\alpha^2(t_1 - \gamma)^{2\beta} - 2 \end{aligned} \right\} \\ > 0 \tag{18}$$

3.2 Fuzzy Model

Due to uncertainty in the environment, it is not easy to define all the parameters precisely, accordingly we assume some of these parameters namely

$\tilde{a}, \tilde{b}, \tilde{p}, \tilde{c}_1, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{h}_1, \tilde{d}, \tilde{s}, \tilde{l}, \tilde{lm}, \tilde{u}$, may change within some limits. Let $\tilde{a} = (a_1, a_2, a_3)$, $\tilde{b} = (b_1, b_2, b_3)$, $\tilde{p} = (p_1, p_2, p_3)$, $\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\tilde{\beta} = (\beta_1, \beta_2, \beta_3)$, $\tilde{c}_1 = (c_{11}, c_{12}, c_{13})$, $\tilde{h}_1 = (h_{11}, h_{12}, h_{13})$, $\tilde{d} = (d_1, d_2, d_3)$, $\tilde{s} = (s_1, s_2, s_3)$, $\tilde{l} = (l_1, l_2, l_3)$, $\tilde{O}_1 = (O_{11}, O_{12}, O_{13})$, $\tilde{lm} = (lm_1, lm_2, lm_3)$, $\tilde{u} = (u_1, u_2, u_3)$ are as triangular fuzzy numbers.

Therefore, the total average cost of the system per unit time in fuzzy sense is given by

$\widetilde{TAC} =$

$$\frac{1}{T} \left[\tilde{c}_1(\tilde{a} - \tilde{b}\tilde{p}) \left\{ t_1 + \frac{\tilde{\alpha}}{\tilde{\beta} + 1} [(t_1 - \tilde{\gamma})^{\tilde{\beta}+1} + \tilde{\gamma}^{\tilde{\beta}+1} - \alpha(-\tilde{\gamma})^{\tilde{\beta}} t_1] + \left(\frac{\tilde{u}^{-\tilde{\lambda}t_1} - \tilde{u}^{-\tilde{\lambda}T}}{\tilde{\lambda} \log \tilde{u}} \right) \right\} + \right. \\ \left. (\tilde{h}_0 + \tilde{h}_1)(\tilde{a} - \tilde{b}\tilde{p}) \left\{ \frac{t_1^2}{2} + \frac{\tilde{\alpha}}{\tilde{\beta} + 1} \left[t_1(t_1 - \tilde{\gamma})^{\tilde{\beta}+1} - \frac{(t_1 - \tilde{\gamma})^{\tilde{\beta}+2}}{\tilde{\beta} + 2} \right] - \frac{\tilde{\alpha}}{(\tilde{\beta} + 1)(\tilde{\beta} + 2)} (t_1 - \tilde{\gamma})^{\tilde{\beta}+2} \right\} \right. \\ \left. + \tilde{d}(\tilde{a} - \tilde{b}\tilde{p}) \left\{ \frac{\tilde{\alpha}}{\tilde{\beta} + 1} [(t_1 - \tilde{\gamma})^{\tilde{\beta}+1} + \tilde{\gamma}^{\tilde{\beta}+1}] - \tilde{\alpha}(-\tilde{\gamma})^{\tilde{\beta}} t_1 \right\} + \tilde{s}(\tilde{a} - \tilde{b}\tilde{p})(T - t_1) \right. \\ \left. + \tilde{l}(\tilde{a} - \tilde{b}\tilde{p}) \left\{ (T - t_1) + \left(\frac{\tilde{u}^{-\tilde{\lambda}t_1} - \tilde{u}^{-\tilde{\lambda}T}}{\tilde{\lambda} \log \tilde{u}} \right) \right\} + (\tilde{O}_0 + \tilde{O}_1) \right] \quad (19)$$

We defuzzify the average total cost $\widetilde{TAC}$ by graded mean representation, signed distance and centroid methods.

3.2.1 Graded Mean Representation Method

Using the graded mean representation method total average cost is given by

$$\widetilde{TAC}_G = \frac{1}{6} [\widetilde{TAC}_{G_1} + 4\widetilde{TAC}_{G_2} + \widetilde{TAC}_{G_3}] \quad (20)$$

where $\widetilde{TAC}_{G_1} =$

$$\frac{1}{T} \left[c_{11}(a_1 - b_1 p_1) \left\{ t_1 + \frac{\alpha_1}{\beta_1 + 1} [(t_1 - \gamma_1)^{\beta_1+1} + \gamma_1^{\beta_1+1} - \alpha_1(-\gamma_1)^{\beta_1} t_1] + \left(\frac{u_1^{-\lambda_1 t_1} - u_1^{-\lambda_1 T}}{\lambda_1 \log u_1} \right) \right\} \right. \\ \left. + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}} \right) (a_1 - b_1 p_1) \left\{ \frac{t_1^2}{2} + \frac{\alpha_1}{\beta_1 + 1} \left[t_1(t_1 - \gamma_1)^{\beta_1+1} - \frac{(t_1 - \gamma_1)^{\beta_1+2}}{\beta_1 + 2} \right] - \frac{\alpha_1}{(\beta_1 + 1)(\beta_1 + 2)} (t_1 - \gamma_1)^{\beta_1+2} \right\} \right. \\ \left. + d_1(a_1 - b_1 p_1) \left\{ \frac{\alpha_1}{\beta_1 + 1} [(t_1 - \gamma_1)^{\beta_1+1} + \gamma_1^{\beta_1+1}] - \alpha_1(-\gamma_1)^{\beta_1} t_1 \right\} + s_1(a_1 - b_1 p_1)(T - t_1) \right. \\ \left. + l_1(a_1 - b_1 p_1) \left\{ (T - t_1) + \left(\frac{u_1^{-\lambda_1 t_1} - u_1^{-\lambda_1 T}}{\lambda_1 \log u_1} \right) \right\} + \left\{ O_0 + \frac{O_{11}}{n^{\Delta_1}} \right\} \right]$$

$$\widehat{TAC}_{G_2} = \left[\frac{1}{T} \left[c_{12}(a_2 - b_2 p_2) \left\{ t_1 + \frac{\alpha_2}{\beta_2 + 1} [(t_1 - \gamma_2)^{\beta_2+1} + \gamma_2^{\beta_2+1} - \alpha_2(-\gamma_2)^{\beta_2} t_1] + \left(\frac{u_2^{-\lambda_2 t_1} - u_2^{-\lambda_2 T}}{\lambda_2 \log u_2} \right) \right\} \right. \right. \\ \left. + \left(h_0 + \frac{h_{12}}{n^{\Delta_2}} \right) (a_2 - b_2 p_2) \left\{ \frac{t_1^2}{2} + \frac{\alpha_2}{\beta_2 + 1} \left[t_1(t_1 - \gamma_2)^{\beta_2+1} - \frac{(t_1 - \gamma_2)^{\beta_2+2}}{\beta_2 + 2} \right] \right. \right. \\ \left. \left. - \frac{\alpha_2}{(\beta_2 + 1)(\beta_2 + 2)} (t_1 - \gamma_2)^{\beta_2+2} \right\} \right. \\ \left. + d_2(a_2 - b_2 p_2) \left\{ \frac{\alpha_2}{\beta_2 + 1} [(t_1 - \gamma_2)^{\beta_2+1} + \gamma_2^{\beta_2+1}] - \alpha_2(-\gamma_2)^{\beta_2} t_1 \right\} + s_2(a_2 - b_2 p_2)(T - t_1) \right. \\ \left. \left. + l_2(a_2 - b_2 p_2) \left\{ (T - t_1) + \left(\frac{u_2^{-\lambda_2 t_1} - u_2^{-\lambda_2 T}}{\lambda_2 \log u_2} \right) \right\} + \left\{ O_0 + \frac{O_{12}}{n^{\Delta_1}} \right\} \right] \right]$$

and

$$\widehat{TAC}_{G_3} = \left[\frac{1}{T} \left[c_{13}(a_3 - b_3 p_3) \left\{ t_1 + \frac{\alpha_3}{\beta_3 + 1} [(t_1 - \gamma_3)^{\beta_3+1} + \gamma_3^{\beta_3+1} - \alpha_3(-\gamma_3)^{\beta_3} t_1] + \left(\frac{u_3^{-\lambda_3 t_1} - u_3^{-\lambda_3 T}}{\lambda_3 \log u_3} \right) \right\} \right. \right. \\ \left. + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}} \right) (a_3 - b_3 p_3) \left\{ \frac{t_1^2}{2} + \frac{\alpha_3}{\beta_3 + 1} \left[t_1(t_1 - \gamma_3)^{\beta_3+1} - \frac{(t_1 - \gamma_3)^{\beta_3+2}}{\beta_3 + 2} \right] \right. \right. \\ \left. \left. - \frac{\alpha_3}{(\beta_3 + 1)(\beta_3 + 2)} (t_1 - \gamma_3)^{\beta_3+2} \right\} \right. \\ \left. + d_3(a_3 - b_3 p_3) \left\{ \frac{\alpha_3}{\beta_3 + 1} [(t_1 - \gamma_3)^{\beta_3+1} + \gamma_3^{\beta_3+1}] - \alpha_3(-\gamma_3)^{\beta_3} t_1 \right\} + s_3(a_3 - b_3 p_3)(T - t_1) \right. \\ \left. \left. + l_3(a_3 - b_3 p_3) \left\{ (T - t_1) + \left(\frac{u_3^{-\lambda_3 t_1} - u_3^{-\lambda_3 T}}{\lambda_3 \log u_3} \right) \right\} + \left\{ O_0 + \frac{O_{13}}{n^{\Delta_1}} \right\} \right] \right]$$

To minimize the total cost function $\widehat{TAC}_G$ per unit time, the optimal value of t_1 can be obtained by solving the equation

$$\frac{\partial \widehat{TAC}_G}{\partial t_1} = 0, \frac{\partial^2 \widehat{TAC}_G}{\partial T^2} > 0, \frac{\partial^2 \widehat{TAC}_G}{\partial t_1 \partial T} > 0 \text{ and } \frac{\partial^2 \widehat{TAC}_G}{\partial T \partial t_1} > 0$$

This equation is equivalent to

$$\begin{aligned}
& \left[\begin{aligned} & c_{11}(a_1 - b_1p_1)\{1 + \alpha_1[(t_1 - \gamma_1)^{\beta_1} - (-\gamma_1)^{\beta_1}] - (u_1^{-\lambda_1 t_1})\} \\ & + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}}\right)(a_1 - b_1p_1)\left\{t_1 + \alpha_1\left[t_1(t_1 - \gamma_1)^{\beta_1} - \frac{1}{\beta_1+1}(t_1 - \gamma_1)^{\beta_1+1}\right]\right\} \\ & + d_1(a_1 - b_1p_1)\{\alpha_1[(t_1 - \gamma_1)^{\beta_1} + \gamma_1]\} - s_1(a_1 - b_1p_1) + l_1(a_1 - b_1p_1)\{u_1^{-\lambda_1 t_1} - 1\} \\ & c_{12}(a_2 - b_2p_2)\{1 + \alpha_2[(t_1 - \gamma_2)^{\beta_2} - (-\gamma_2)^{\beta_2} - (u_2^{-\lambda_2 t_1})\} + \\ & \left(h_0 + \frac{h_{12}}{n^{\Delta_2}}\right)(a_2 - b_2p_2)\left\{t_1 + \alpha_2\left[t_1(t_1 - \gamma_2)^{\beta_2} - \frac{1}{\beta_2+1}(t_1 - \gamma_2)^{\beta_2+1}\right]\right\} \\ & + d_2(a_2 - b_2p_2)\{\alpha_2[(t_1 - \gamma_2)^{\beta_2} + \gamma_2]\} - s_2(a_2 - b_2p_2) + l_2(a_2 - b_2p_2)\{u_2^{-\lambda_2 t_1} - 1\} \\ & + c_{13}(a_3 - b_3p_3)\{1 + \alpha_3[(t_1 - \gamma_3)^{\beta_3} - (-\gamma_3)^{\beta_3}] - (u_3^{-\lambda_3 t_1})\} \\ & + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}}\right)(a_3 - b_3p_3)\left\{t_1 + \alpha_3\left[t_1(t_1 - \gamma_3)^{\beta_3} - \frac{1}{\beta_3+1}(t_1 - \gamma_3)^{\beta_3+1}\right]\right\} + \\ & d_3(a_3 - b_3p_3)\{\alpha_3[(t_1 - \gamma_3)^{\beta_3} + \gamma_3]\} - s_3(a_3 - b_3p_3) + l_3(a_3 - b_3p_3)\{u_3^{-\lambda_3 t_1} - 1\} \end{aligned} \right] \\
& = 0 \tag{21}
\end{aligned}$$

For the total average cost $\widehat{TAC}_G$ to be convex, then $\frac{\partial^2 \widehat{TAC}_G}{\partial t_1^2} > 0$

Now,

$$\begin{aligned}
& \frac{\partial^2 \widehat{TAC}_G}{\partial t_1^2} = \frac{1}{6T} \left[\begin{aligned} & c_{11}(a_1 - b_1p_1)\{\alpha_1\beta_1[(t_1 - \gamma_1)^{\beta_1-1} - (-\gamma_1)^{\beta_1-1}] + \lambda_1 \log(u_1)(u_1^{-\lambda_1 t_1})\} \\ & + \left(h_0 + \frac{h_1}{n^{\Delta_2}}\right)(a_1 - b_1p_1)\{1 + \alpha_1\beta_1 t_1(t_1 - \gamma_1)^{\beta_1-1}\} + \\ & d_1(a_1 - b_1p_1)\{\alpha_1\beta_1[(t_1 - \gamma_1)^{\beta_1-1}]\} - l_1(a_1 - b_1p_1)\{\lambda_1 \log(u_1)(u_1^{-\lambda_1 t_1})\} \\ & \left\{ c_{12}(a_2 - b_2p_2)\{\alpha_2\beta_2[(t_1 - \gamma_2)^{\beta_2-1} - (-\gamma_2)^{\beta_2-1}] + \lambda_2 \log(u_2)(u_2^{-\lambda_2 t_1})\} \right. \\ & + \left(h_0 + \frac{h_{12}}{n^{\Delta_2}}\right)(a_2 - b_2p_2)\{1 + \alpha_2\beta_2 t_1(t_1 - \gamma_2)^{\beta_2-1}\} + \\ & \left. \left\{ d_2(a_2 - b_2p_2)\{\alpha_2\beta_2(t_1 - \gamma_2)^{\beta_2-1}\} - l_2(a_2 - b_2p_2)\{\lambda_2 \log(u_2)(u_2^{-\lambda_2 t_1})\} \right\} \right. \\ & + c_{13}(a_3 - b_3p_3)\{\alpha_3\beta_3[(t_1 - \gamma_3)^{\beta_3-1} - (-\gamma_3)^{\beta_3-1}] + \lambda_3 \log(u_3)(u_3^{-\lambda_3 t_1})\} \\ & + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}}\right)(a_3 - b_3p_3)\{1 + \alpha_3\beta_3 t_1(t_1 - \gamma_3)^{\beta_3-1}\} + \\ & \left. d_3(a_3 - b_3p_3)\{\alpha_3\beta_3(t_1 - \gamma_3)^{\beta_3-1}\} - l_3(a_3 - b_3p_3)\{\lambda_3 \log(u_3)(u_3^{-\lambda_3 t_1})\} \right] \\ & > 0 \tag{22}
\end{aligned}$$

3.2.2 Signed Distance Method

Using the Signed Distance Method, the total average cost is given by

$$\widehat{TAC}_S = \frac{1}{4} [\widehat{TAC}_{S_1} + 2\widehat{TAC}_{S_2} + \widehat{TAC}_{S_3}] \tag{23}$$

Where

$$\widetilde{TAC}_{S_1} = \frac{1}{T} \left[\begin{aligned} & c_{11}(a_1 - b_1 p_1) \left\{ t_1 + \frac{\alpha_1}{\beta_1 + 1} [(t_1 - \gamma_1)^{\beta_1 + 1} + \gamma_1^{\beta_1 + 1} - \alpha_1 (-\gamma_1)^{\beta_1} t_1] + \left(\frac{u_1^{-\lambda_1 t_1} - u_1^{-\lambda_1 T}}{\lambda_1 \log u_1} \right) \right\} \\ & + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}} \right) (a_1 - b_1 p_1) \left\{ \frac{t_1^2}{2} + \frac{\alpha_1}{\beta_1 + 1} \left[t_1 (t_1 - \gamma_1)^{\beta_1 + 1} - \frac{(t_1 - \gamma_1)^{\beta_1 + 2}}{\beta_1 + 2} \right] \right. \\ & \quad \left. - \frac{\tilde{\alpha}}{(\beta_1 + 1)(\beta_1 + 2)} (t_1 - \gamma_1)^{\beta_1 + 2} \right\} \\ & + d_1(a_1 - b_1 p_1) \left\{ \frac{\alpha_1}{\beta_1 + 1} [(t_1 - \gamma_1)^{\beta_1 + 1} + \gamma_1^{\beta_1 + 1}] - \alpha_1 (-\gamma_1)^{\beta_1} t_1 \right\} + s_1(a_1 - b_1 p_1)(T - t_1) \\ & l_1(a_1 - b_1 p_1) \left\{ (T - t_1) + \left(\frac{u_1^{-\lambda_1 t_1} - u_1^{-\lambda_1 T}}{\lambda_1 \log u_1} \right) \right\} + \left\{ O_0 + \frac{O_{11}}{n^{\Delta_1}} \right\} \end{aligned} \right],$$

$$\widetilde{TAC}_{S_2} = \frac{1}{T} \left[\begin{aligned} & c_{12}(a_2 - b_2 p_2) \left\{ t_1 + \frac{\alpha_2}{\beta_2 + 1} [(t_1 - \gamma_2)^{\beta_2 + 1} + \gamma_2^{\beta_2 + 1} - \alpha_2 (-\gamma_2)^{\beta_2} t_1] + \left(\frac{u_2^{-\lambda_2 t_1} - u_2^{-\lambda_2 T}}{\lambda_2 \log u_2} \right) \right\} \\ & + \left(h_0 + \frac{h_{12}}{n^{\Delta_2}} \right) (a_2 - b_2 p_2) \left\{ \frac{t_1^2}{2} + \frac{\alpha_2}{\beta_2 + 1} \left[t_1 (t_1 - \gamma_2)^{\beta_2 + 1} - \frac{(t_1 - \gamma_2)^{\beta_2 + 2}}{\beta_2 + 2} \right] \right. \\ & \quad \left. - \frac{\alpha_2}{(\beta_2 + 1)(\beta_2 + 2)} (t_1 - \gamma_2)^{\beta_2 + 2} \right\} \\ & + d_2(a_2 - b_2 p_2) \left\{ \frac{\alpha_2}{\beta_2 + 1} [(t_1 - \gamma_2)^{\beta_2 + 1} + \gamma_2^{\beta_2 + 1}] - \alpha_2 (-\gamma_2)^{\beta_2} t_1 \right\} + s_2(a_2 - b_2 p_2)(T - t_1) \\ & l_2(a_2 - b_2 p_2) \left\{ (T - t_1) + \left(\frac{u_2^{-\lambda_2 t_1} - u_2^{-\lambda_2 T}}{\lambda_2 \log u_2} \right) \right\} + \left\{ O_0 + \frac{O_{12}}{n^{\Delta_1}} \right\} \end{aligned} \right]$$

And

$$\widetilde{TAC}_{S_3} =$$

$$\frac{1}{T} \left[c_{13}(a_3 - b_3 p_3) \left\{ t_1 + \frac{\alpha_3}{\beta_3 + 1} [(t_1 - \gamma_3)^{\beta_3+1} + \gamma_3^{\beta_3+1} - \alpha_3(-\gamma_3)^{\beta_3} t_1] + \left(\frac{u_3^{-\lambda_3 t_1} - u_3^{-\lambda_3 T}}{\lambda_3 \log u_3} \right) \right\} \right. \\ \left. + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}} \right) (a_3 - b_3 p_3) \left\{ \frac{t_1^2}{2} + \frac{\alpha_3}{\beta_3 + 1} \left[t_1(t_1 - \gamma_3)^{\beta_3+1} - \frac{(t_1 - \gamma_3)^{\beta_3+2}}{\beta_3 + 2} \right] \right. \right. \\ \left. \left. - \frac{\alpha_3}{(\beta_3 + 1)(\beta_3 + 2)} (t_1 - \gamma_3)^{\beta_3+2} \right\} \right. \\ \left. + d_3(a_3 - b_3 p_3) \left\{ \frac{\alpha_3}{\beta_3 + 1} [(t_1 - \gamma_3)^{\beta_3+1} + \gamma_3^{\beta_3+1}] - \alpha_3(-\gamma_3)^{\beta_3} t_1 \right\} + s_3(a_3 - b_3 p_3)(T - t_1) \right. \\ \left. + l_3(a_3 - b_3 p_3) \left\{ (T - t_1) + \left(\frac{u_3^{-\lambda_3 t_1} - u_3^{-\lambda_3 T}}{\lambda_3 \log u_3} \right) \right\} + \left\{ O_0 + \frac{O_{13}}{n^{\Delta_1}} \right\} \right]$$

To minimize the total average cost function $\widehat{TAC}_S$, the optimal value of t_1 can be obtained by solving the equation

$$\frac{\partial \widehat{TAC}_S}{\partial t_1} = 0 \quad \frac{\partial \widehat{TAC}_G}{\partial T} = 0 \text{ which is equivalent to}$$

$$\frac{1}{4T} \left[2 \left\{ \begin{aligned} & c_{11}(a_1 - b_1 p_1) \{ 1 + \alpha_1 [(t_1 - \gamma_1)^{\beta_1} - (-\gamma_1)^{\beta_1}] - (u_1^{-\lambda_1 t_1}) \} \\ & + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}} \right) (a_1 - b_1 p_1) \left\{ t_1 + \alpha_1 \left[t_1(t_1 - \gamma_1)^{\beta_1} - \frac{1}{\beta_1 + 1} (t_1 - \gamma_1)^{\beta_1+1} \right] \right\} \\ & + d_1(a_1 - b_1 p_1) \{ \alpha_1 [(t_1 - \gamma_1)^{\beta_1} + \gamma_1] \} - s_1(a_1 - b_1 p_1) + l_1(a_1 - b_1 p_1) \{ u_1^{-\lambda_1 t_1} - 1 \} \\ & c_{12}(a_2 - b_2 p_2) \{ 1 + \alpha_2 [(t_1 - \gamma_2)^{\beta_2} - (-\gamma_2)^{\beta_2} - (u_2^{-\lambda_2 t_1}) \} + \\ & \left(h_0 + \frac{h_{12}}{n^{\Delta_2}} \right) (a_2 - b_2 p_2) \left\{ t_1 + \alpha_2 \left[t_1(t_1 - \gamma_2)^{\beta_2} - \frac{1}{\beta_2 + 1} (t_1 - \gamma_2)^{\beta_2+1} \right] \right\} \\ & + d_2(a_2 - b_2 p_2) \{ \alpha_2 [(t_1 - \gamma_2)^{\beta_2} + \gamma_2] \} - s_2(a_2 - b_2 p_2) + l_2(a_2 - b_2 p_2) \{ u_2^{-\lambda_2 t_1} - 1 \} \\ & + c_{13}(a_3 - b_3 p_3) \{ 1 + \alpha_3 [(t_1 - \gamma_3)^{\beta_3} - (-\gamma_3)^{\beta_3}] - (u_3^{-\lambda_3 t_1}) \} \\ & + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}} \right) (a_3 - b_3 p_3) \left\{ t_1 + \alpha_3 \left[t_1(t_1 - \gamma_3)^{\beta_3} - \frac{1}{\beta_3 + 1} (t_1 - \gamma_3)^{\beta_3+1} \right] \right\} + \\ & d_3(a_3 - b_3 p_3) \{ \alpha_3 [(t_1 - \gamma_3)^{\beta_3} + \gamma_3] \} - s_3(a_3 - b_3 p_3) + l_3(a_3 - b_3 p_3) \{ u_3^{-\lambda_3 t_1} - 1 \} \end{aligned} \right\} \right]$$

$$= 0 \quad (24)$$

For the total average cost function $\widehat{TAC}_S$ to be convex, the following condition must be satisfied

$$\frac{\partial^2 \widehat{TAC}_S}{\partial t_1^2} > 0$$

Now,

$$\frac{\partial^2 \widehat{TAC}_S}{\partial t_1^2} = \frac{1}{4T} + 2 \left[\begin{aligned} & c_{11}(a_1 - b_1 p_1) \{ \alpha_1 \beta_1 [(t_1 - \gamma_1)^{\beta_1 - 1} - (-\gamma_1)^{\beta_1 - 1}] + \lambda_1 \log(u_1)(u_1^{-\lambda_1 t_1}) \} \\ & + \left(h_0 + \frac{h_1}{n^{\Delta_2}} \right) (a_1 - b_1 p_1) \{ 1 + \alpha_1 \beta_1 t_1 (t_1 - \gamma_1)^{\beta_1 - 1} \} + \\ & d_1(a_1 - b_1 p_1) \{ \alpha_1 \beta_1 [(t_1 - \gamma_1)^{\beta_1 - 1}] - l_1(a_1 - b_1 p_1) \{ \lambda_1 \log(u_1)(u_1^{-\lambda_1 t_1}) \} \\ & \left\{ c_{12}(a_2 - b_2 p_2) \{ \alpha_2 \beta_2 [(t_1 - \gamma_2)^{\beta_2 - 1} - (-\gamma_2)^{\beta_2 - 1}] + \lambda_2 \log(u_2)(u_2^{-\lambda_2 t_1}) \} \right. \\ & + \left(h_0 + \frac{h_{12}}{n^{\Delta_2}} \right) (a_2 - b_2 p_2) \{ 1 + \alpha_2 \beta_2 t_1 (t_1 - \gamma_2)^{\beta_2 - 1} \} + \\ & \left. \left\{ d_2(a_2 - b_2 p_2) \{ \alpha_2 \beta_2 (t_1 - \gamma_2)^{\beta_2 - 1} \} - l_2(a_2 - b_2 p_2) \{ \lambda_2 \log(u_2)(u_2^{-\lambda_2 t_1}) \} \right\} \right. \\ & + c_{13}(a_3 - b_3 p_3) \{ \alpha_3 \beta_3 [(t_1 - \gamma_3)^{\beta_3 - 1} - (-\gamma_3)^{\beta_3 - 1}] + \lambda_3 \log(u_3)(u_3^{-\lambda_3 t_1}) \} \\ & + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}} \right) (a_3 - b_3 p_3) \{ 1 + \alpha_3 \beta_3 t_1 (t_1 - \gamma_3)^{\beta_3 - 1} \} + \\ & \left. d_3(a_3 - b_3 p_3) \{ \alpha_3 \beta_3 (t_1 - \gamma_3)^{\beta_3 - 1} \} - l_3(a_3 - b_3 p_3) \{ \lambda_3 \log(u_3)(u_3^{-\lambda_3 t_1}) \} \right] \\ & > 0 \end{aligned} \right] \quad (25)$$

3.2.3 Centroid Method

Using the Centroid Method, the total average cost function is given by

$$\widehat{TAC}_C = \frac{1}{3} [\widehat{TAC}_{C_1} + \widehat{TAC}_{C_2} + \widehat{TAC}_{C_3}] \quad (26)$$

Where

$$\widehat{TAC}_{C_1} = \frac{1}{T} \left[\begin{aligned} & c_{11}(a_1 - b_1 p_1) \left\{ t_1 + \frac{\alpha_1}{\beta_1 + 1} [(t_1 - \gamma_1)^{\beta_1 + 1} + \gamma_1^{\beta_1 + 1} - \alpha_1 (-\gamma_1)^{\beta_1} t_1] + \left(\frac{u_1^{-\lambda_1 t_1} - u_1^{-\lambda_1 T}}{\lambda_1 \log u_1} \right) \right\} \\ & + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}} \right) (a_1 - b_1 p_1) \left\{ \frac{t_1^2}{2} + \frac{\alpha_1}{\beta_1 + 1} \left[t_1 (t_1 - \gamma_1)^{\beta_1 + 1} - \frac{(t_1 - \gamma_1)^{\beta_1 + 2}}{\beta_1 + 2} \right] \right. \\ & \quad \left. - \frac{\tilde{\alpha}}{(\beta_1 + 1)(\beta_1 + 2)} (t_1 - \gamma_1)^{\beta_1 + 2} \right\} \\ & + d_1(a_1 - b_1 p_1) \left\{ \frac{\alpha_1}{\beta_1 + 1} [(t_1 - \gamma_1)^{\beta_1 + 1} + \gamma_1^{\beta_1 + 1}] - \alpha_1 (-\gamma_1)^{\beta_1} t_1 \right\} + s_1(a_1 - b_1 p_1)(T - t_1) \\ & l_1(a_1 - b_1 p_1) \left\{ (T - t_1) + \left(\frac{u_1^{-\lambda_1 t_1} - u_1^{-\lambda_1 T}}{\lambda_1 \log u_1} \right) \right\} + \left\{ O_0 + \frac{O_{11}}{n^{\Delta_1}} \right\} \end{aligned} \right],$$

$$\widehat{TAC}_{C_2} =$$

$$\frac{1}{T} \left[c_{12}(a_2 - b_2 p_2) \left\{ t_1 + \frac{\alpha_2}{\beta_2 + 1} [(t_1 - \gamma_2)^{\beta_2+1} + \gamma_2^{\beta_2+1} - \alpha_2(-\gamma_2)^{\beta_2} t_1] + \left(\frac{u_2^{-\lambda_2 t_1} - u_2^{-\lambda_2 T}}{\lambda_2 \log u_2} \right) \right\} \right. \\ \left. + \left(h_0 + \frac{h_{12}}{n^{\Delta_2}} \right) (a_2 - b_2 p_2) \left\{ \frac{t_1^2}{2} + \frac{\alpha_2}{\beta_2 + 1} \left[t_1(t_1 - \gamma_2)^{\beta_2+1} - \frac{(t_1 - \gamma_2)^{\beta_2+2}}{\beta_2 + 2} \right] \right. \right. \\ \left. \left. - \frac{\alpha_2}{(\beta_2 + 1)(\beta_2 + 2)} (t_1 - \gamma_2)^{\beta_2+2} \right\} \right. \\ \left. + d_2(a_2 - b_2 p_2) \left\{ \frac{\alpha_2}{\beta_2 + 1} [(t_1 - \gamma_2)^{\beta_2+1} + \gamma_2^{\beta_2+1}] - \alpha_2(-\gamma_2)^{\beta_2} t_1 \right\} + s_2(a_2 - b_2 p_2)(T - t_1) \right. \\ \left. + l_2(a_2 - b_2 p_2) \left\{ (T - t_1) + \left(\frac{u_2^{-\lambda_2 t_1} - u_2^{-\lambda_2 T}}{\lambda_2 \log u_2} \right) \right\} + \left\{ O_0 + \frac{O_{12}}{n^{\Delta_1}} \right\} \right]$$

and

$$\widehat{TAC}_{C_3} =$$

$$\frac{1}{T} \left[c_{13}(a_3 - b_3 p_3) \left\{ t_1 + \frac{\alpha_3}{\beta_3 + 1} [(t_1 - \gamma_3)^{\beta_3+1} + \gamma_3^{\beta_3+1} - \alpha_3(-\gamma_3)^{\beta_3} t_1] + \left(\frac{u_3^{-\lambda_3 t_1} - u_3^{-\lambda_3 T}}{\lambda_3 \log u_3} \right) \right\} \right. \\ \left. + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}} \right) (a_3 - b_3 p_3) \left\{ \frac{t_1^2}{2} + \frac{\alpha_3}{\beta_3 + 1} \left[t_1(t_1 - \gamma_3)^{\beta_3+1} - \frac{(t_1 - \gamma_3)^{\beta_3+2}}{\beta_3 + 2} \right] \right. \right. \\ \left. \left. - \frac{\alpha_3}{(\beta_3 + 1)(\beta_3 + 2)} (t_1 - \gamma_3)^{\beta_3+2} \right\} \right. \\ \left. + d_3(a_3 - b_3 p_3) \left\{ \frac{\alpha_3}{\beta_3 + 1} [(t_1 - \gamma_3)^{\beta_3+1} + \gamma_3^{\beta_3+1}] - \alpha_3(-\gamma_3)^{\beta_3} t_1 \right\} + s_3(a_3 - b_3 p_3)(T - t_1) \right. \\ \left. + l_3(a_3 - b_3 p_3) \left\{ (T - t_1) + \left(\frac{u_3^{-\lambda_3 t_1} - u_3^{-\lambda_3 T}}{\lambda_3 \log u_3} \right) \right\} + \left\{ O_0 + \frac{O_{13}}{n^{\Delta_1}} \right\} \right]$$

To minimize the total average cost function $\widehat{TAC}_C$, the optimal value of t_1 can be obtained by solving the equation

$$\frac{\partial \widehat{TAC}_C}{\partial t_1} = 0$$

which is equivalent to

$$\begin{aligned}
& \left[\begin{aligned} & c_{11}(a_1 - b_1p_1)\{1 + \alpha_1[(t_1 - \gamma_1)^{\beta_1} - (-\gamma_1)^{\beta_1}] - (u_1^{-\lambda_1 t_1})\} \\ & + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}}\right)(a_1 - b_1p_1)\left\{t_1 + \alpha_1\left[t_1(t_1 - \gamma_1)^{\beta_1} - \frac{1}{\beta_1 + 1}(t_1 - \gamma_1)^{\beta_1 + 1}\right]\right\} \\ & + d_1(a_1 - b_1p_1)\{\alpha_1[(t_1 - \gamma_1)^{\beta_1} + \gamma_1]\} - s_1(a_1 - b_1p_1) + l_1(a_1 - b_1p_1)\{u_1^{-\lambda_1 t_1} - 1\} \\ & c_{12}(a_2 - b_2p_2)\{1 + \alpha_2[(t_1 - \gamma_2)^{\beta_2}] - (-\gamma_2)^{\beta_2} - (u_2^{-\lambda_2 t_1})\} + \\ & \left(h_0 + \frac{h_{12}}{n^{\Delta_2}}\right)(a_2 - b_2p_2)\left\{t_1 + \alpha_2\left[t_1(t_1 - \gamma_2)^{\beta_2} - \frac{1}{\beta_2 + 1}(t_1 - \gamma_2)^{\beta_2 + 1}\right]\right\} \\ & + d_2(a_2 - b_2p_2)\{\alpha_2[(t_1 - \gamma_2)^{\beta_2} + \gamma_2]\} - s_2(a_2 - b_2p_2) + l_2(a_2 - b_2p_2)\{u_2^{-\lambda_2 t_1} - 1\} \\ & + c_{13}(a_3 - b_3p_3)\{1 + \alpha_3[(t_1 - \gamma_3)^{\beta_3} - (-\gamma_3)^{\beta_3}] - (u_3^{-\lambda_3 t_1})\} \\ & + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}}\right)(a_3 - b_3p_3)\left\{t_1 + \alpha_3\left[t_1(t_1 - \gamma_3)^{\beta_3} - \frac{1}{\beta_3 + 1}(t_1 - \gamma_3)^{\beta_3 + 1}\right]\right\} + \\ & d_3(a_3 - b_3p_3)\{\alpha_3[(t_1 - \gamma_3)^{\beta_3} + \gamma_3]\} - s_3(a_3 - b_3p_3) + l_3(a_3 - b_3p_3)\{u_3^{-\lambda_3 t_1} - 1\} \end{aligned} \right] \\
& = 0 \tag{27}
\end{aligned}$$

For the total average cost function $\widetilde{TAC}_C$ to be convex, the following condition must be satisfied

$$\frac{\partial^2 \widetilde{TAC}_C}{\partial t_1^2} > 0,$$

Now,

$$\begin{aligned}
\frac{\partial^2 \widetilde{TAC}_C}{\partial t_1^2} = \frac{1}{3T} + & \left[\begin{aligned} & c_{11}(a_1 - b_1p_1)\{\alpha_1\beta_1[(t_1 - \gamma_1)^{\beta_1 - 1} - (-\gamma_1)^{\beta_1 - 1}] + \lambda_1 \log(u_1)(u_1^{-\lambda_1 t_1})\} \\ & + \left(h_0 + \frac{h_{11}}{n^{\Delta_2}}\right)(a_1 - b_1p_1)\{1 + \alpha_1\beta_1 t_1(t_1 - \gamma_1)^{\beta_1 - 1}\} + \\ & d_1(a_1 - b_1p_1)\{\alpha_1\beta_1[(t_1 - \gamma_1)^{\beta_1 - 1}]\} - l_1(a_1 - b_1p_1)\{\lambda_1 \log(u_1)(u_1^{-\lambda_1 t_1})\} \\ & \left\{ c_{12}(a_2 - b_2p_2)\{\alpha_2\beta_2[(t_1 - \gamma_2)^{\beta_2 - 1} - (-\gamma_2)^{\beta_2 - 1}] + \lambda_2 \log(u_2)(u_2^{-\lambda_2 t_1})\} \right. \\ & + \left(h_0 + \frac{h_{12}}{n^{\Delta_2}}\right)(a_2 - b_2p_2)\{1 + \alpha_2\beta_2 t_1(t_1 - \gamma_2)^{\beta_2 - 1}\} + \\ & \left. \left\{ d_2(a_2 - b_2p_2)\{\alpha_2\beta_2(t_1 - \gamma_2)^{\beta_2 - 1}\} - l_2(a_2 - b_2p_2)\{\lambda_2 \log(u_2)(u_2^{-\lambda_2 t_1})\} \right\} \right. \\ & + c_{13}(a_3 - b_3p_3)\{\alpha_3\beta_3[(t_1 - \gamma_3)^{\beta_3 - 1} - (-\gamma_3)^{\beta_3 - 1}] + \lambda_3 \log(u_3)(u_3^{-\lambda_3 t_1})\} \\ & + \left(h_0 + \frac{h_{13}}{n^{\Delta_2}}\right)(a_3 - b_3p_3)\{1 + \alpha_3\beta_3 t_1(t_1 - \gamma_3)^{\beta_3 - 1}\} + \\ & \left. d_3(a_3 - b_3p_3)\{\alpha_3\beta_3(t_1 - \gamma_3)^{\beta_3 - 1}\} - l_3(a_3 - b_3p_3)\{\lambda_3 \log(u_3)(u_3^{-\lambda_3 t_1})\} \right] > 0 \tag{28}
\end{aligned}$$

1.3 Numerical Example

Let $T = 30$, $a = 350$ units, $b = 0.2$, $p = \text{₦}20/\text{unit}$, $\gamma = 0.8$, $\alpha = 0.01$, $\beta = 1.2$, $\lambda = 1.5$, $c_1 = \text{₦}10/\text{unit}$, $s = \text{₦}5/\text{unit}$, $O_0 = \text{₦}150/\text{order}$, $O_1 = \text{₦}100$, $h_0 = \text{₦}0.3/\text{unit}$, $h_1 = \text{₦}0.2/\text{unit}$, $d = \text{₦}11/\text{unit}$, $l = \text{₦}6/\text{unit}$, $u = 1.2$, $n = 3$, $\Delta_1 = 0.5$, $\Delta_2 = 0.2$

Results of the crisp model

The optimal values of t_1 and TAC. were determined using MATLAB software as:

$t_1^* = 3.511$ days and $TAC^* = \text{N}3957.67$ respectively and the corresponding ordering quantity

$Q^* = 1654.6$ units

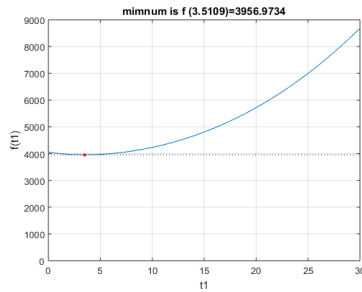


Fig. 2 Graph of T.A.C. against t_1

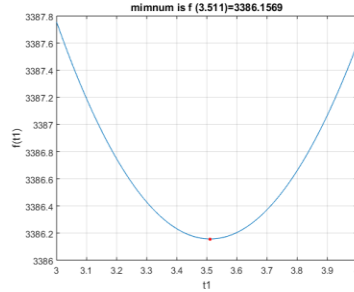


Fig. 3 Convexity graph of TAC

1.3.1 Fuzzy model

Let $\tilde{a} = (200, 300, 400)$, $\tilde{b} = (0.1, 0.2, 0.3)$, $\tilde{p} = (10, 20, 30)$, $\tilde{\gamma} = (0.6, 0.8, 1.0)$, $\tilde{\alpha} = (0.005, 0.01, 0.015)$, $\tilde{\beta} = (1.0, 1.2, 1.4)$, $\tilde{c}_1 = (6, 10, 14)$, $\tilde{h}_1 = (0.1, 0.2, 0.3)$, $\tilde{d} = (8, 11, 14)$, $\tilde{s} = (4, 5, 6)$, $\tilde{l} = (4, 6, 8)$, $\tilde{o}_1 = (50, 100, 150)$, $\tilde{lm} = (1.3, 1.5, 1.7)$, and $\tilde{u} = (1.1, 1.2, 1.3)$

Using MATLAB software, the optimal values of t_1^* , TAC^* and Q^* for the respective methods are as follows:

1. Graded Mean Representation $t_1^* = 2.9815$ days, $TAC^* = \text{N}3280.77$ and the corresponding ordering quantity $Q^* = 1552.9$ units.
2. Signed Distance Method $t_1^* = 2.7778$ days and $TAC^* = \text{N}3226.88$ and the corresponding ordering quantity $Q^* = 1518.3$ units.
3. Centroid Method $t_1^* = 2.6029$ days and $TAC^* = \text{N}3172.42$ and the corresponding ordering quantity $Q^* = 1490.1$ units.

These are shown below in figure 4 (a), (b) and (c) respectively.

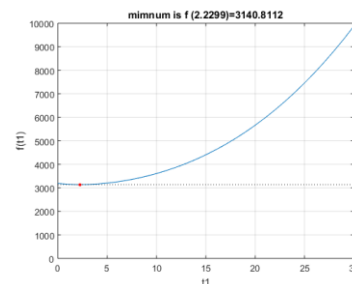
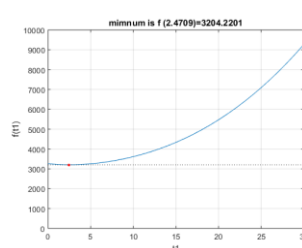
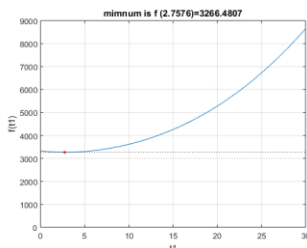


Fig. 4 (a) (b) (c)

3.4 Sensitivity Analysis

1. From table 1-3 below it is observed that, by increasing the demand parameter $\tilde{a}$ and keeping all other parameters constant, fuzzy total average cost TAC_G , TAC_S and TAC_C increases together with the order quantity Q and t_1 for all the three methods.
2. An increase in the parameter $\tilde{b}$, results in an increase in t_1 , while TAC_G , TAC_S , TAC_C and Q decreases for all the methods.
3. Increasing the parameter $\tilde{p}$ and keeping other parameters constant, results in an increase in t_1 whereas TAC_G , TAC_S , TAC_C and Q decreases for both methods.
4. As $\tilde{a}$ increases, the total average cost TAC_G , TAC_S , TAC_C and order quantity Q also increases while t_1 decreases for all the methods.
5. An increase in the parameter $\tilde{\beta}$ results in a decrease in both total average cost, t_1 and order quantity Q .
6. Increasing the backlogging parameter $\tilde{lm}$ and keeping other parameters constant results in a decrease in both TAC_G , TAC_S , TAC_C , t_1 and Q .
7. An increase in $\tilde{u}$ and keeping other parameters constant results in a decrease for both the total average cost, t_1 as well as the order quantity Q .

Table 1: Graded Mean Representation Method

		t_1	TAC	Q
$\tilde{a}$	(100, 200, 300)	2.6507	2179.61	1003.18
	(200, 300, 400)	2.7576	3266.48	1515.00
	(300, 400, 500)	2.8139	4353.25	2026.83
$\tilde{b}$	(0.1, 0.15, 0.2)	2.7546	3279.69	1520.12
	(0.15, 0.2, 0.25)	2.7550	3268.58	1515.00
	(0.2, 0.25, 0.3)	2.7554	3257.48	1509.88
$\tilde{p}$	(10, 15, 20)	2.7546	3279.69	1520.12
	(15, 20, 25)	2.7550	3268.58	1515.00
	(20, 25, 30)	2.7554	3257.48	1509.88
$\tilde{a}$	(0.004, 0.007, 0.01)	2.9332	3263.22	1512.38
	(0.007, 0.01, 0.013)	2.7793	3266.08	1515.00
	(0.01, 0.013, 0.016)	2.6474	3268.65	1517.62
$\tilde{\beta}$	(0.8, 1.0, 1.2)	2.8018	3268.50	1523.47
	(1.0, 1.2, 1.4)	2.7576	3266.48	1515.00
	(1.2, 1.4, 1.6)	2.7138	3264.58	1507.03
$\tilde{lm}$	(0.9, 1.2, 1.5)	2.9749	3297.06	1744.28
	(1.2, 1.5, 1.8)	2.7307	3265.98	1515.00
	(1.5, 1.8, 2.1)	2.5318	3245.65	1507.03
$\tilde{u}$	(1.1, 1.2, 1.3)	2.7576	3266.48	1515.00
	(1.2, 1.3, 1.4)	2.3809	3228.94	1260.24

	(1.3, 1.4, 1.5)	2.1280	3211.87	1151.97
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Table 2: Signed Distance Method

		t_1	TAC	Q
$\tilde{a}$	(100, 200, 300)	2.3423	2145.09	973.14
	(200, 300, 400)	2.4709	3204.22	1469.63
	(300, 400, 500)	2.5404	4263.20	1966.13
$\tilde{b}$	(0.1, 0.15, 0.2)	2.4674	3218.30	1474.60
	(0.15, 0.2, 0.25)	2.4679	3207.36	1469.63
	(0.2, 0.25, 0.3)	2.4684	3196.42	1464.67
$\tilde{p}$	(10, 15, 20)	2.4674	3218.30	1474.60
	(15, 20, 25)	2.4679	3207.36	1469.63
	(20, 25, 30)	2.4684	3196.41	1464.67
$\tilde{\alpha}$	(0.004, 0.007, 0.01)	2.6168	3201.44	1467.62
	(0.007, 0.01, 0.013)	2.4953	3203.75	1469.63
	(0.01, 0.013, 0.016)	2.3895	3205.85	1471.65
$\tilde{\beta}$	(0.8, 1.0, 1.2)	2.4943	3206.03	1478.26
	(1.0, 1.2, 1.4)	2.4709	3204.22	1469.63
	(1.2, 1.4, 1.6)	2.4477	3202.52	1461.48
$\tilde{m}$	(0.9, 1.2, 1.5)	2.6683	3235.93	1705.30
	(1.2, 1.5, 1.8)	2.4334	3203.29	1469.63
	(1.5, 1.8, 2.1)	2.2420	3181.62	1320.09
$\tilde{u}$	(1.1, 1.2, 1.3)	2.4709	3204.22	1469.63
	(1.2, 1.3, 1.4)	2.1176	3163.72	1203.69
	(1.3, 1.4, 1.5)	1.8783	3145.11	1087.78

Table 3: Centroid Method

		t_1	TAC	Q
$\tilde{a}$	(100, 200, 300)	2.0914	2109.67	949.78
	(200, 300, 400)	2.2299	3140.81	1434.36
	(300, 400, 500)	2.3065	4171.76	1918.94
$\tilde{b}$	(0.1, 0.15, 0.2)	2.2264	3155.75	1439.21
	(0.15, 0.2, 0.25)	2.2269	3144.98	1434.36
	(0.2, 0.25, 0.3)	2.2274	3134.21	1429.51
$\tilde{p}$	(10, 15, 20)	2.2264	3155.75	1439.21
	(15, 20, 25)	2.2269	3144.98	1434.36

	(20, 25, 30)	2.2274	3134.21	1429.51
$\tilde{\alpha}$	(0.004, 0.007, 0.01)	2.3512	3138.39	1432.78
	(0.007, 0.01, 0.013)	2.2547	3140.30	1434.36
	(0.01, 0.013, 0.016)	2.1694	3142.06	1435.94
	(0.01, 0.013, 0.016)	2.1694	3142.06	1435.94
$\tilde{\beta}$	(0.8, 1.0, 1.2)	2.2397	3142.34	1443.04
	(1.0, 1.2, 1.4)	2.2299	3140.81	1434.36
	(1.2, 1.4, 1.6)	2.2202	3139.38	1426.12
$\tilde{lm}$	(0.9, 1.2, 1.5)	2.4104	3173.53	1675.17
	(1.2, 1.5, 1.8)	2.1842	3139.36	1434.36
	(1.5, 1.8, 2.1)	2.0001	3116.37	1280.30
$\tilde{u}$	(1.1, 1.2, 1.3)	2.2299	3140.81	1434.36
	(1.2, 1.3, 1.4)	1.8984	3097.47	1159.20
	(1.3, 1.4, 1.5)	1.6729	3077.40	1036.79

3.5 Conclusion

In real life situation, some items such as drugs, chemicals, vegetables etc. deteriorates with the passage of time in storage facilities. Therefore, due to uncertainty, deterioration was considered in this model as a triangular fuzzy number. The demand function is also not constant and keeps changing depending on the selling price of items. Hence, it was also considered as a triangular fuzzy number in the model. Other cost components that were considered as triangular fuzzy numbers include the ordering, holding, shortage, lost sale costs. The graded mean integration representation, signed distance, and centroid methods were applied in defuzzifying the total average cost. The model was developed using both crisp and fuzzy approach and it was observed that the fuzzy total average cost was less as compared to the crisp model. It was also observed that the centroid method of defuzzification gives the minimum total average cost, followed by the signed distance method and the graded mean representation method respectively.

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