



The Simpson's $-\frac{3}{8}$ Type Hybrid Block Method for Solving Stiff Systems of Ordinary Differential Equations

Lawrence Osa Adoghe

Department of Mathematics, Ambrose Alli University, Ekpoma, Nigeria

Email address: adoghelarry@gmail.com, adolaw@aauekpoma.edu.ng

Abstract

A hybrid Simpson's $\frac{3}{8}$ -type second derivative method is proposed in this paper for the solution of stiff ordinary differential equations. The method is derived by interpolation and collocation of power series to generate continuous linear multistep methods (clmms). The derived schemes consist of the main method and two additional methods which are combined to form the block method. The analysis of the properties of the methods shows that the method, besides being consistent and zero-stable, is also A-stable; which is a reliable property for solving stiff ordinary differential equations.

Keywords Hybrid Simpson's $-\frac{3}{8}$ rule, Block method, Second derivative, A-stable, Stiff Systems, Zero-Stability.

1. Introduction

Consider the first order ordinary differential equation given by

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (1)$$

where $x \in [x_0, x_N]$ and $f: [x_0, x_N] \times R^m \rightarrow R^m$ is continuous and differentiable and f satisfies the Lipschitz condition (Henrici 1962)

The equation (1) is said to be stiff if the following conditions hold

(i) $\text{Re}(\lambda_i) < 0, \quad i = 1, 2, \dots, m$

(ii) $\max_i |\text{Re}(\lambda_i)| \gg \min_i |\text{Re}(\lambda_i)|, \quad i = 1, 2, \dots, m.$

where (λ_i) 's are the eigenvalues of the Jacobian matrix. The stiff ratio is given as

$$\max_i |\text{Re}(\lambda_i)| \gg \min_i |\text{Re}(\lambda_i)|, \quad i = 1, 2, \dots, m$$

The conventional K-step LMM for solving (1) numerically is given by

$$\sum_{j=0}^k a_j y_{n+j} = h \sum_{j=0}^k b_j y'_{n+j} \quad (2)$$

Method (2) has $2k + 1$ unknowns a 's and b 's and is of order $2k$, where k is the step number

A lot of important improvements or modifications have been made in the class of LMM in (2) above. These modifications are done in order to improve the accuracy and ensure wide region

of absolute stability of the method. These modifications included the use of higher derivatives and inclusion of off-grid points or hybrid points (Ezzeddine and Hojjati (2012)). Most of these modifications resulted in the Backward Differentiation Formula (BDF), the hybrid multistep methods, etc

The modification of (2) has also resulted in the introduction of second and third derivative methods. These are more advanced methods of (2) where second and third derivatives of the solution are used. A lot of work has been done in this direction by several authors. These include the works of Enright(1974) who developed second derivative multistep method for solving stiff ordinary differential equations, Cash(1981) developed second derivative extended backward differentiation formula for the solution of stiff systems. Also Ngwane and Jator (2012) used block hybrid second derivative method of order six with one hybrid point to solve stiff systems of ODEs. Other second derivative methods developed for solving stiff systems are Adewale et al(2018), Akinfenwa et al(2017), Sabo et al (2019). Adewale et al (2018), developed an A- stable second derivative method with varying step-length for solving stiff systems. Also developed is a Third Derivative Multistep for solving Stiff Systems by Ezzeddine and Hojjati (2012). Recently, Akinfenwa et al (2017) developed Simpson's $-\frac{3}{8}$ type block second derivative multistep method for the solution of stiff ordinary differential equations. Following Akinfenwa et al (2017), and the logic behind Simpson's $-\frac{3}{8}$ rule for quadrature, we propose a one-step block hybrid second derivative method with two hybrid points for solving systems of ordinary differential equations. The proposed method shall be made up of one main method and two additional methods which are combined to form the block matrix.

2. Materials and Methods.

The Simpson's $-\frac{3}{8}$ formula is given by

$$\int_{x_0}^{x_0+3h} f(x)dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] \quad (3)$$

Akinfenwa et al (2017), modified (3) and developed Simpson's $\frac{3}{8}$ -type block second derivative multistep method of the form:

$$y(n+k) - y_n = h \sum_{j=0}^k b_j f_{n+j} + h^2 \sum_{j=0}^k g_j g_{n+j} \quad (4)$$

where $k=1,2,3$ and b_j, g_j are the coefficients.

We shall proceed to derive our proposed method by introducing hybrid points. The proposed method is a modification of (4) where $k = 1$

Suppose the exact solution of one of (1) is $Y(x)$ in the range $[x_n, x_{n+h}]$, we shall represent the approximate solution (1) by the polynomial basis function of the form:

$$Y(x) = \sum_{j=0}^{r+s-1} a_j y_j(x) \quad (5)$$

where the a_j 's are the coefficient to be determined and $y_j(x) = x^j, j = 0,1,2,...,8$ is a polynomial function of degree $r + 2s - 1$, with r and s as the number of interpolation and

collocation points respectively. We shall construct our method by imposing the following conditions on (5). Thus, we shall let

$$Y(x_n) = y_n \quad (6)$$

$$Y'(x) = \sum_{j=1}^{r+2s-1} a_j y(x) = f_{n+j} \quad (7)$$

$$Y''(x) = \sum_{j=2}^{r+2s-1} j(j-1)a_j y(x) = g_{n+j} \quad (8)$$

where $f_{n+j} = f(x_{n+j}, y(x_{n+j}))$

$$f_{n+jv} = f(x_{n+jv}, y(x_{n+jv}))$$

$$g_{n+j} = \left. \frac{df(x, y(x))}{dx} \right|_{y_{n+j}}^{x_{n+j}}$$

$$g_{n+jv} = \left. \frac{df(x, y(x))}{dx} \right|_{y_{n+jv}}^{x_{n+jv}}$$

Equations (6), (7) and (8) lead to a system of equations of the form

$$AX = B \quad (9)$$

Solving (9) for the coefficients a_j 's and substituting in (5) and after much algebraic manipulation, we obtained the following continuous scheme

$$Y(t) = a_0(t)y_n + h \sum_{j=0,1}^{\infty} b_j(t)f_{n+j} + \sum_{v=\frac{1}{3}, \frac{2}{3}}^{\infty} b_v(t)f_{n+vj} + h^2 \sum_{j=0,1}^{\infty} d_j(t)g_{n+j} + \sum_{v=\frac{1}{3}, \frac{2}{3}}^{\infty} d_v(t)g_{n+vj} \quad (10)$$

where $b_0, b_1, b_v, d_0, d_1, d_v$ are the continuous coefficients given below as

$a_0(t)$	0	0	0	0	0	0	0	0
$b_0(t)$	1	0	$-\frac{97}{4}$	$\frac{1691}{16}$	$-\frac{423}{2}$	$\frac{903}{4}$	$-\frac{3483}{28}$	$\frac{891}{32}$
$b_{\frac{1}{3}}(t)$	0	0	0	$\frac{243}{4}$	$-\frac{243}{8}$	$\frac{2997}{8}$	$-\frac{3645}{14}$	$\frac{2187}{32}$
$b_{\frac{2}{3}}(t)$	0	0	$\frac{81}{4}$	$-\frac{2187}{16}$	$\frac{729}{2}$	$-\frac{1863}{4}$	$\frac{8019}{28}$	$-\frac{2187}{32}$
$b_1(t)$	0	0	4	$-\frac{119}{4}$	90	$-\frac{1077}{8}$	$\frac{1377}{14}$	$-\frac{891}{32}$

$d_0(t)$	h^0	0	$\frac{1}{2}$	$-\frac{11}{3}$	$\frac{193}{16}$	$-\frac{108}{5}$	$\frac{87}{4}$	$-\frac{81}{7}$	$\frac{81}{32}$
$d_{1/3}(t)$	h^1	0	0	-9	54	$-\frac{2619}{20}$	$\frac{1269}{8}$	$-\frac{2673}{28}$	$\frac{729}{32}$
$d_{2/3}(t)$	h^2	0	0	$-\frac{9}{2}$	$\frac{513}{16}$	$-\frac{459}{5}$	$\frac{513}{4}$	$-\frac{1215}{14}$	$\frac{729}{32}$
$d_1(t)$	h^3	0	0	$-\frac{1}{3}$	$\frac{5}{2}$	$-\frac{153}{20}$	$\frac{93}{8}$	$-\frac{243}{28}$	$\frac{81}{32}$

By evaluating (10) at $x = x_{n+1}$ and letting $v = \frac{1}{3}$ and $\frac{2}{3}$, we have one main method and two additional methods as given below:

$$y_{n+1} = y_n + \frac{h}{224} [81f_n + 31f_{n+\frac{1}{3}} + 31f_{n+\frac{2}{3}} + 81f_{n+1}] + \frac{h^2}{3360} [19g_n - 27g_{n+\frac{1}{3}} + 27g_{n+\frac{2}{3}} - 19g_{n+1}] \quad (11)$$

$$y_{n+\frac{1}{3}} = y_n + \frac{h}{54432} [6893f_n + 8451f_{n+\frac{1}{3}} + 2403f_{n+\frac{2}{3}} + 397f_{n+1}] + \frac{h^2}{272160} [1283g_n - 7659g_{n+\frac{1}{3}} - 2421g_{n+\frac{2}{3}} - 1467g_{n+1}] \quad (12)$$

$$y_{n+\frac{2}{3}} = y_n + \frac{h}{1701} [223f_n + 540f_{n+\frac{1}{3}} + 351f_{n+\frac{2}{3}} + 20f_{n+1}] + \frac{h^2}{8505} [43g_n - 144g_{n+\frac{1}{3}} - 171g_{n+\frac{2}{3}} - 8g_{n+1}] \quad (13)$$

Equations 11-13 are then combined to form the block.

2.2. Order and error constant

The Taylor series expansion of methods (11),(12) and (13) yield

$$\begin{aligned} & \left[\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j (h)^j}{j!} y_n^j - y_n - \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{(j+1)!} y_n^{j+1} \cdot \left[\frac{6893}{54432} (0)^{j-1} + \frac{313}{2016} \left(\frac{1}{3}\right)^{j-1} \right. \right. \\ & \quad + \frac{89}{2016} \left(\frac{1}{3}\right)^{j-1} + \frac{397}{54432} (1)^{j-1} \left. \right] - \sum_{j=0}^{\infty} \frac{(h)^{j+2}}{(j+2)!} y_n^{j+2} \cdot \left[\frac{1283}{272160} (0)^{j-2} \right. \\ & \quad - \frac{851}{30240} \left(\frac{1}{3}\right)^{j-2} - \frac{269}{30240} \left(\frac{2}{3}\right)^{j-2} - \frac{163}{272160} (1)^{j-2} \left. \right] = 0 \\ & \left[\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j (h)^j}{j!} y_n^j - y_n - \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{(j+1)!} y_n^{j+1} \cdot \left[\frac{223}{1701} (0)^{j-1} + \frac{20}{63} \left(\frac{1}{3}\right)^{j-1} + \frac{13}{63} \left(\frac{2}{3}\right)^{j-1} \right. \right. \\ & \quad + \frac{20}{1701} (1)^{j-1} \left. \right] - \sum_{j=0}^{\infty} \frac{(h)^{j+2}}{(j+2)!} y_n^{j+2} \cdot \left[\frac{43}{8505} (0)^{j-2} - \frac{16}{945} \left(\frac{1}{3}\right)^{j-2} - \frac{19}{945} \left(\frac{2}{3}\right)^{j-2} \right. \\ & \quad - \frac{8}{8505} (1)^{j-2} \left. \right] = 0 \\ & \left[\sum_{j=0}^{\infty} \frac{(1)^j (h)^j}{j!} y_n^j - y_n - \sum_{j=0}^{\infty} \frac{(h)^{j+1}}{(j+1)!} y_n^{j+1} \cdot \left[\frac{31}{224} (0)^{j-1} + \frac{81}{224} \left(\frac{1}{3}\right)^{j-1} + \frac{81}{224} \left(\frac{2}{3}\right)^{j-1} \right. \right. \\ & \quad + \frac{31}{224} (1)^{j-1} \left. \right] - \sum_{j=0}^{\infty} \frac{(h)^{j+2}}{(j+2)!} y_n^{j+2} \cdot \left[\frac{19}{3360} (0)^{j-2} - \frac{9}{1120} \left(\frac{1}{3}\right)^{j-2} + \frac{9}{1120} \left(\frac{2}{3}\right)^{j-2} \right. \\ & \quad - \frac{19}{3360} (1)^{j-2} \left. \right] = 0 \end{aligned} \quad (14)$$

Thus the analysis of methods shows they are of order $p = [8, 8, 8]^T$ with error constants given as $C_{p+1} = [4.9997 \times 10^{-11}, 8.3203 \times 10^{-10}, 1.4581 \times 10^{-9}]^T$

2.3 Stability Analysis

The block matrix finite difference form of the methods in (11-13) can be written as

$$A^{(0)}Y_a = A^{(1)}Y_{a-1} + hB^{(0)}F_a + B^{(1)}F_{a-1} + h^2C^{(0)}G_a + C^{(1)}G_{a-1} \quad (15)$$

$$\text{where } Y_a = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix}^T, Y_{a-1} = \begin{bmatrix} y_{n-\frac{1}{3}} \\ y_{n-\frac{2}{3}} \\ y_n \end{bmatrix}^T, F_a = \begin{bmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{bmatrix}^T$$

$$F_{a-1} = \begin{bmatrix} f_{n-\frac{1}{3}} \\ f_{n-\frac{2}{3}} \\ f_n \end{bmatrix}^T, G_a = \begin{bmatrix} g_{n+\frac{1}{3}} \\ g_{n+\frac{2}{3}} \\ g_{n+1} \end{bmatrix}^T, G_{a-1} = \begin{bmatrix} g_{n-\frac{1}{3}} \\ g_{n-\frac{2}{3}} \\ g_n \end{bmatrix}^T$$

$$\text{And } A^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} \frac{313}{2016} & \frac{89}{2016} & \frac{397}{54432} \\ \frac{20}{63} & \frac{13}{63} & \frac{20}{1701} \\ \frac{81}{224} & \frac{81}{224} & \frac{31}{224} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 0 & \frac{6893}{54432} \\ 0 & \frac{223}{1701} \\ 0 & \frac{31}{224} \end{bmatrix}, C^{(0)} = \begin{bmatrix} \frac{851}{30240} & -\frac{269}{30240} & -\frac{163}{272160} \\ \frac{16}{945} & -\frac{19}{945} & \frac{-8}{8505} \\ \frac{9}{1120} & \frac{9}{1120} & \frac{19}{3360} \end{bmatrix}$$

2.4 Zero stability

The zero stability property is concerned with the stability of difference system in (15) in the limit as $h \rightarrow 0$. Thus (15) becomes

$$A^{(0)}Y_a - A^{(1)}Y_{a-1} = 0 \quad (16)$$

The characteristic polynomial of the above is given by

$$r(l) = \det(lA^{(0)} - A^{(1)}) \quad (17)$$

$$\text{Thus } r(l) = \det(lA^{(0)} - A^{(1)}) = \det \begin{bmatrix} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{bmatrix} = l^3(l-1)$$

The block method (15) is zero stable since from (17); if $r(l) = 0$ satisfies $|l_i| \leq 1$, $i = 1, 2, 3$ and for $|l_i| = 1$, the multiplicity does not exceed 1

2.5 Consistency

The method (15) is consistent as it has order $p > 1$, hence according to Henrici (1962), we can say that our method is convergent.

2.6 Linear stability

In the spirit of Brugano and Triagiante(1998),and Hairer and Wanner(1996), we shall consider the following test equations

$$y' = l y, y'' = l^2 y, \text{ where } l \text{ is a constant}$$

When applied to (15), we have

$$Y_a = M(z)Y_{a-1}, z = l h \quad (18)$$

Where the matrix $M(z)$ is given as

$$M(z) = (A^{(0)} - zB^{(0)} - z^2C^{(0)})^{-1}(A^{(1)} + zB^{(1)} + z^2C^{(1)}) \quad (19)$$

From (15), we shall obtain the stability polynomial $R(z)$ given as

$$R(z) = \frac{z^6 + 33z^5 + 579z^4 + 6480z^3 + 46980z^2 + 204120z + 480240}{z^6 - 33z^5 + 579z^4 - 6480z^3 + 46980z^2 - 204120z + 480240} \quad (20)$$

The region of absolute stability of the method is therefore defined as

$$G = \{z \in \mathbb{C} : |R(z)| \leq 1\} \quad (21)$$

The stability requirement of the method for solving stiff system is that the method is A-stable. For method (15), this can only be achieved if the left – half complex plane is contained in G.

The plot representing the stability region corresponding to the stability function $R(z)$ is given below

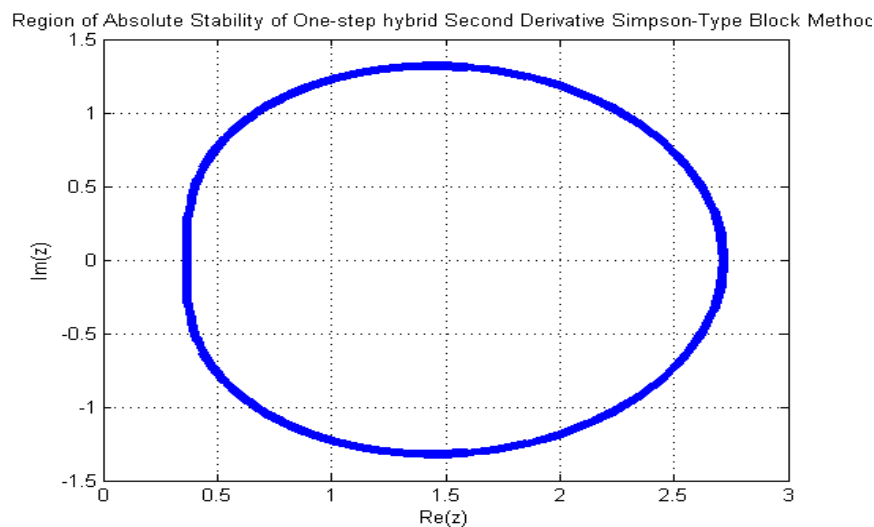


Fig.1. Region of Absolute stability of hybrid second derivative method

2.7 Numerical Experiments

The method shall be implemented by combining the methods (11),(12)and (13) as block simultaneous integrators without the requirements for starting values. Thus; when $n = 0$ the values $(y_{1/3}, y_{2/3}, y_1)$ are obtained simultaneously in the sub interval (x_0, x_1) given that y_0 is known. Also when $n = 1$, we have $(y_{4/3}, y_{5/3}, y_2)$ simultaneously over the subinterval (x_1, x_2) and subsequently the values $(y_{7/3}, y_{8/3}, y_3), (y_{10/3}, y_{11/3}, y_4), \dots$ over the sub-interval

Problem 1

We consider the linear stiff problem

$$\begin{aligned} y' &= -y + 95z, y(0) = 1 \\ z' &= -y - 97z, z(0) = 1 \\ x &\in [0, 1] \end{aligned}$$

Which has the exact solution

$$\begin{aligned} y(x) &= \frac{1}{47} (95e^{-2x} - 48e^{-96x}) \\ z(x) &= \frac{1}{47} (48e^{-96x} - e^{-2x}) \end{aligned}$$

Source: Abhulimen and Ukpebor (2019)

Table 1

H	X	Method	Computed sol./error $y(x)$ ($ error $)	Computed sol./error $z(x)$ ($ error $)
$\frac{1}{8}$	10	$\frac{3}{8}$ method	0.27355004(1.61313×10^{-10})	- 0.0028794741(1.688×10^{-12})
$\frac{1}{16}$	10	Abhulimen&Ukpebor(2019)	0.27354997(3.0×10^{-8})	- 0.002879474(1.4×10^{-9})
			0.27354656(7.8×10^{-5})	- 0.0029794811(3.7×10^{-5})
		Abhulimen&Otunta (2006)	0.27355004(1.91×10^{-9})	- 0.002879474(2.01×10^{-9})
			0.036994834(2.73×10^{-1})	0.00038942(2.49×10^{-9})
		BDF1	0.27355004(2.1444×10^{-10})	- 0.0028794741(2.518×10^{-12})
		BDF2		
		$\frac{3}{8}$ method		

$\frac{1}{32}$	10	Abhulimen&Ukpebor Abhulimen&Otunta BDF1 BDF2 $\frac{3}{8}$ method	0.27354997($3.0' 10^{-8}$) 0.27354656($7.8' 10^{-5}$) 0.27355004($1.91' 10^{-9}$) 0.036994834($2.73' 10^{-1}$) 0.27355004($5.0475' 10^{-10}$)	- 0.002879474($1.4' 10^{-9}$) - 0.0029794811($3.7' 10^{-5}$) - 0.002879474($2.01' 10^{-9}$) 0.00038942($2.49' 10^{-9}$) - 0.00287947410($6.3005' 10^{-12}$)
	10	Exact solution	0.273550040584643	- 0.00287947410542666

Problem 2

$$y_1 = 198y_1 + 199y_2, \quad y_1(0) = 1, h = 0.1$$

$$y_1 = -398y_1 + 399y_2, \quad y_2(0) = -1, h = -1$$

Assume $h = 0.1$ in the numerical solution

This has the exact solution

$$y_1(x) = e^{-x}, y_2(x) = -e^{-x}$$

Source: Sabo *et al* (2019)

Table 2

x- value	Exact solution		Computed solution	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	0.9048374180	- 0.9048374180	0.9048374221	- 0.9048374231
0.2	0.8187307531	- 0.8187307531	0.8187307596	- 0.8187307605
0.3	0.7408182207	- 0.7408182207	0.7408182290	- 0.7408182299
0.4	0.6703200460	- 0.6703200460	0.6703200558	- 0.6703200566
0.5	0.6065306597	- 0.6065306597	0.6065306706	- 0.6065306713
0.6	0.5488116361	- 0.5488116361	0.5488116478	- 0.5488116485
0.7	0.4965853038	- 0.4965853038	0.4965853161	- 0.4965853167
0.8	0.4493289641	- 0.4493289641	0.4493289768	- 0.4493289773
0.9	0.4065696597	- 0.4065696597	0.4065696726	- 0.4065696730
1.0	0.3678794412	- 0.3678794412	0.3678794541	- 0.3678794544

Table 3

X- value	Error in Sabo et al(2019)		Error in the New Method	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	2.60×10^{-6}	2.60×10^{-6}	4.06×10^{-9}	5.06×10^{-9}
0.2	2.42×10^{-6}	2.42×10^{-6}	6.49×10^{-9}	7.44×10^{-9}
0.3	1.18×10^{-6}	1.18×10^{-6}	8.39×10^{-9}	9.24×10^{-9}
0.4	3.90×10^{-6}	3.90×10^{-6}	9.86×10^{-9}	1.06×10^{-8}
0.5	5.58×10^{-6}	5.58×10^{-6}	1.10×10^{-8}	1.17×10^{-8}
0.6	3.23×10^{-6}	3.23×10^{-6}	1.18×10^{-8}	1.24×10^{-8}
0.7	4.35×10^{-6}	4.35×10^{-6}	1.24×10^{-8}	1.29×10^{-8}
0.8	3.97×10^{-6}	3.97×10^{-6}	1.27×10^{-8}	1.32×10^{-9}
0.9	3.59×10^{-6}	3.59×10^{-6}	1.29×10^{-8}	1.33×10^{-8}
1.0	4.31×10^{-6}	4.30×10^{-6}	1.29×10^{-8}	1.33×10^{-8}

Problem 3:

$$y'' + 1001y' + 1000y = 0, \quad y(0) = 1, y'(0) = -1$$

$$y(x) = e^{-x}$$

Source: Abhulimen and Ukpebor (2019) and Adeniran and Longe(2019)

Table 4

Step	x-value	Method	Computed solution / absolute error $y(x) / (error)$
0.1	10	Adeniran& Longe(2019) $\frac{3}{8}$ method	$0.367879420284596 (2.088685 \times 10^{-8})$ $0.367879441320468 (1.4903 \times 10^{-10})$
0.05	10	Abhulimen&Okunuga(2008) Abhulimen& Omeike(2011) Abhulimen&Ukpebor(2019) $\frac{3}{8}$ method	$0.36787930 (1.8 \times 10^{-7})$ $0.36787846 (1.4 \times 10^{-8})$ $0.367879450 (8.9 \times 10^{-8})$ $0.367879441150831 (2.0611 \times 10^{-11})$
		Exact solution	0.367879441171442

Problem 4

$$y_1' = 998y_1 + 1998y_2, \quad y_1(0) = 1$$

$$y_2' = -999y_1 - 1999y_2, \quad y_2(0) = 0, \quad h = 0.01$$

exact solution $y_1(x) = 2e^{-x} - e^{-1000x}$, $y_2(x) = e^{-x} - e^{-1000x}$

Source: Sabo *et al* (2019)

Table 5

x-value	Exact solution		Computed solution	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	1.9800542675	- 0.9900952336	1.976935753	-0.9868859183
0.2	1.9603973445	- 0.9801986753	1.9603873386	- 0.8187307605
0.3	1.9408910670	-0.9704455335	1.9408910369	-0.9704455033
0.4	1.9215788783	-0.9607894392	1.9215788788	0.9607894400
0.5	1.9024588490	-0.9512294245	1.9024588487	-0.9512294250
0.6	1.8835290672	-0.9417645338	1.8835290660	-0.9417645336
0.7	1.8647876398	-0.9323938199	1.8647876378	-0.9323938196
0.8	1.8462326928	-0.9231163464	1.8462326900	-0.9231163457
0.9	1.8278623705	-0.9139311853	1.8278623679	-0.9139311841
1.0	1.8096748361	-0.9048374180	1.8096748317	-0.9048374164

Table 6

X-value	Error in Sabo et al(2019)		Error in the New Method	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	$2.60' 10^{-6}$	$2.60' 10^{-6}$	$3.1185' 10^{-3}$	$3.1115' 10^{-3}$
0.2	$2.42' 10^{-6}$	$2.42' 10^{-6}$	$1.0005' 10^{-5}$	$1.0006' 10^{-5}$
0.3	$1.18' 10^{-6}$	$1.18' 10^{-6}$	$3.01835' 10^{-8}$	$3.02393' 10^{-8}$
0.4	$3.90' 10^{-6}$	$3.90' 10^{-6}$	$5.0157' 10^{-10}$	$8.82178' 10^{-10}$
0.5	$5.58' 10^{-6}$	$5.58' 10^{-6}$	$2.6567' 10^{-10}$	$5.4072' 10^{-10}$
0.6	$3.23' 10^{-6}$	$3.23' 10^{-6}$	$1.1192' 10^{-9}$	$1.0839' 10^{-10}$
0.7	$4.35' 10^{-6}$	$4.35' 10^{-6}$	$1.9540' 10^{-9}$	$3.1569' 10^{-10}$
0.8	$3.97' 10^{-6}$	$3.97' 10^{-6}$	$2.7722' 10^{-9}$	$7.3135' 10^{-10}$
0.9	$3.59' 10^{-6}$	$3.59' 10^{-6}$	$3.5739' 10^{-9}$	$1.1387' 10^{-9}$
1.0	$4.31' 10^{-6}$	$4.30' 10^{-6}$	$4.3594' 10^{-9}$	$1.5378' 10^{-9}$

Problem 5

$$y_1' = -100y_1 + 9.901y_2, \quad y_1(0) = 1$$

$$y_2' = 0.1y_1 - y_2, \quad y_2(0) = 10$$

Exact solution $y_1(x) = 2e^{0.99x}$, $y_2(x) = 10e^{-0.99x}$

Source: Sabo *et al* (2018)

Table 7

x-value	Exact solution		Computed solution	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	0.905742708023548	9.05742708023548	0.905742708093	9.0574270761883
0.2	0.820369853137831	8.20369853137831	0.820369852835709	8.20369852404733
0.3	0.743044012361940	7.43044012361940	0.743044011756321	7.43044011365962
0.4	0.673006695937386	6.73006695937386	0.673006695088163	6.73006694734599
0.5	0.609570907296309	6.09570907296309	0.609570906254786	6.09570905934548
0.6	0.552114404306931	5.52114404306931	0.552114403116904	5.52114402826850
0.7	0.500073595695768	5.00073595695768	0.500073594394486	5.00073594131771
0.8	0.468134327348097	4.52938012776558	0.452938011395565	4.52938011157614
0.9	0.410245302259044	4.10245302259044	0.410245300824929	4.10245300609407
1.0	0.371576691022046	3.71576691022046	0.371576689557093	3.71576689361884

Table 8

X-value	Error in Sabo et al(2018)		Error in the New Method	
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	$5.00' \cdot 10^{-10}$	$4.00' \cdot 10^{-9}$	$6.945' \cdot 10^{-11}$	$4.047' \cdot 10^{-9}$
0.2	$1.50' \cdot 10^{-9}$	$7.00' \cdot 10^{-9}$	$3.021' \cdot 10^{-10}$	$7.330' \cdot 10^{-9}$
0.3	$1.50' \cdot 10^{-9}$	$9.00' \cdot 10^{-9}$	$6.056' \cdot 10^{-10}$	$9.959' \cdot 10^{-9}$
0.4	$2.20' \cdot 10^{-9}$	$1.30' \cdot 10^{-8}$	$8.492' \cdot 10^{-10}$	$1.202' \cdot 10^{-8}$
0.5	$2.20' \cdot 10^{-9}$	$1.50' \cdot 10^{-8}$	$1.041' \cdot 10^{-9}$	$1.361' \cdot 10^{-8}$
0.6	$2.30' \cdot 10^{-9}$	$1.30' \cdot 10^{-8}$	$1.90' \cdot 10^{-9}$	$1.480' \cdot 10^{-8}$
0.7	$1.30' \cdot 10^{-9}$	$1.20' \cdot 10^{-8}$	$1.301' \cdot 10^{-9}$	$1.563' \cdot 10^{-8}$
0.8	$2.10' \cdot 10^{-8}$	$1.40' \cdot 10^{-8}$	$1.380' \cdot 10^{-9}$	$1.618' \cdot 10^{-8}$
0.9	$1.70' \cdot 10^{-9}$	$1.50' \cdot 10^{-8}$	$1.434' \cdot 10^{-9}$	$1.649' \cdot 10^{-8}$
1.0	$2.40' \cdot 10^{-9}$	$1.70' \cdot 10^{-8}$	$1.464' \cdot 10^{-9}$	$1.660' \cdot 10^{-8}$

3.0 Results and Discussion

Table 1 Shows the Comparative analysis of results of problem 1 with methods of Abhulimen and Ukpebor(2019), Abhulimen and Otunta(2006)and Sofoluwe et al (2013) BDF(1&2). The accuracy of the proposed method was compared with those of Abhulimen and Ukpebor (2019), Abhulimen and Otunta(2008) and BDF(1&2) at various step length $\left(\frac{1}{8}, \frac{1}{16}\right)$. At the point $x = 10$ the new method was seen to have performed better. Table 3 shows the comparison of errors in problem 2 using the proposed method with errors in Sabo et al (2019). It is shown that our new method is more accurate than the methods proposed by Sabo, et al (2019). Table 4 shows the results of problem 3 compared with Abhulimen and Ukpebor (201 and Adeniran and Longe(2019). In it our proposed method performed better than the methods of Adeniran and Longe(2019) , Abhulimen and Okunuga(2008), Abhulimen and Omeike(2011) and Abhulimen and Ukpebor(2019). In Table 6, Errors in problem 4 using our proposed method compared with errors in Sabo et al(2019). Our proposed method performed more accurately than that of Sabo et al (2019) and in Table 8 the accuracy of our proposed method was compared with those of Sabo et al(2018) in problem 5. The new method was found to have performed better in both cases.

4. Conclusion

A hybrid Simpson $-\frac{3}{8}$ type second derivative block method has been developed. The method is found to have good stability properties appropriate for solving stiff systems. The accuracy of the method was tested on some stiff systems. The results shows that the method is effective and compared favorably with some existing methods in the literature.

References

- [1].Abhulimen, C.E. and Otunta, F.O.(2006): A Sixth Order Multiderivative Multistep Methods for Stiff Systems of Differential Equations. International Journal Numerical Mathematics(IJNM), Vol.2(1),248-268
- [2]. Abhulimen, C.E. and Omeike, G.E. (2011). A Sixth order exponentially fitted scheme for the numerical solution of system of ordinary differential equations. Journal of Applied Mathematics and Bioinformatics,1 (1), 175-186
- [3]. Abhulimen, C.E. and Okunuga,S.A. (2008) Exponentially fitted second derivative method for Stiff initial value problems in ODEs. Journal of Engineering Science and Applications, 5, 36-47
- [4]. Sofoluwe ,A.B., Okunuga, S.A. and Ehijie, J.O.(2013) Composite Multiderivative Multistep Backward Differential Formula (BDF1&2) for solving stiff initial value problems, Journal of Mathematics Sciences, NMC, Abuja, 2(1), 227-239
- [5]. Adeniran, A. O and .Longe, I. O (2019). Solving Directly Second order Initial Value Problems with Lucas Polynomial. Journal of Advances in Mathematics and Computer Science, 32(4):1-7
- [6]. Ali. K.Ezzeddine and Gholamreza Hojjati(2012). Third Derivative Multistep for Stiff Systems. International Journal of Nonlinear Science, vol.14(4), 441-450
- [7]. Abhulimen C.E, and Ukpebor, L.A (2019).A new class of second derivative for numerical solution of stiff initial value problems .Journal of the Mathematical Society of Nigeria, vol.38(2), 259-270

- [8]. Cash. J.R. (1981). Second derivative extended backward differentiation formulas for the integration of the stiff systems, SIAM J. of Numer. Anal., 18, 21-36
- [9]. Enright, W.H.(1974). Second derivatives methods for stiff ordinary differential equations. SIAM J. Num.Anal. ; 11(2);321-331
- [10].Hairer, E.and Wanner, G (1996), Solving Ordinary Differential Equations II, Springer, New York.
- [11]. Ngwane, F.F and Samuel Jato (2012) Block hybrid-second Derivative Method for stiff systems. Inter. Journal of Pure and Applied Mathematics,80(4),543-559
- [12]. James Adewale, Adesanya Olaide, Onsachi Oziohu, Sunday Joshua, Moses Omuya1. (2018). Proficiency of Second Derivative Schemes for the Numerical Solution of Stiff Systems. American Journal of Computational Mathematics,8,96-107.<https://doi.org/10.4236/ajcm.2018.81008>
- [13]. Brugnano, D. Trigiante, Solving Differential Problems by Multistep Initial and Boundary value Methods, Gordon and Breach Science Publishers, Amsterdam (1998).
- [14]. Akinfenwa, O.A; Abdulganiy, R.I. Okunuga, S.A and Irechukwu, V. (2017), Simpson's $-\frac{3}{8}$ type block method for stiff system of ordinary differential equations, 36(3); 503-514
- [15] Henrici, P. Discrete Variable Methods in ODEs, John Wiley, New York (1962).
- [16]. Sabo John; Yusuf, T.Kyaga and Adamu, A. Bumbur (2019). Second Derivative Two-Step Block Hybrid Enright's Linear Multistep Methods for Solving Initial Value Problems of General Second Order Stiff Ordinary Differential Equations. Journal Of Advances in Mathematics And Computer Science. 30(2);1-10
- [17]. Sabo John, Pius Tumba, Bakari, A. I. (2018): The Modification of Second Derivative Linear Multistep Ordinary Differential Equation for solving stiffly Differential Equation. Volume 14, Issue 5, 48-55.