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Stability of Motions Around Non-Collinear Points in The Photogravitational Er3bp: Impact of a Circumbinary Disc on Tv Crateris

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Abstract

In this communication, the effect of an additional gravitational potential from a circumbinary disc of gaseous dust particles on the non-collinear points of the photogravitational elliptic restricted three-body problem is examined. The triangular points of the problem when both luminous primaries move in elliptic orbits around their common barycenter and are surrounded by a circumbinary disc are found to exhibit the peculiar property of shifting towards the origin, with a decrease in the radiation pressure factors q_1 and q_2 towards the line joining the primaries with increase in the potential from the disc. In an important finding, the added perturbation due to the disc is found to have a stabilizing effect, as is evident from the critical mass parameter μ_c. The binary system (HD 98800 B) TV Crateris in the constellation Crater which has both stellar primaries surrounded by a disc provides a perfect model for this dynamic system.

Keywords: Elliptic Restricted Three-Body Problem, Binary Systems, Celestial Mechanics, Photogravitational, Potential from a disc

1. Introduction

The general three-body problem formulated and studied by Newton in 1687 in his *principia* requires that future and past motions of three astronomical bodies be determined uniquely based solely on their present positions and velocities and there are no restrictions on their masses or on the initial conditions. If one of the masses of the three astronomical bodies becomes very negligible when compared to the masses of the rest, the general three-body problem becomes the restricted three-body problem, and there are two possibilities, namely the two bodies with dominant masses move around their center of mass either along circular or elliptic orbits, which lead to the circular or elliptic restricted three-body problems respectively. The three-body problem has been studied for over three centuries. The orbits of celestial bodies are mostly elliptic. It is therefore more accurate to study the ER3BP since it generalizes the original CR3BP and improves its application, while some outstanding and useful properties of the circular model still hold true or can be adapted to the elliptic case. This has prompted many researchers from [20-24] to study the ER3BP. Not quite long, [10] have investigated in the framework of the ER3BP the motion of a dust particle, in an orbit around triangular equilibrium points when both oblate and luminous primaries move around their common center of mass respectively. There are circular discs of dust particles in the cosmos which are observed to share a common similarity to the Edgeworth-Kuiper belt [1], [6], [11], [14-16]. In their research [17] considered the equatorial plane coinciding with the plane of motion and one of the primaries as an oblate spheroid. They [18] extended the generalized elliptic restricted three body problem for [17]. In their study [19] studied the stability of L_4 where both the infinitesimal mass and one of the primaries have been taken as oblate spheroids. Many researchers including [2-5], [7], [9] were prompted to study the CR3BP by taking into cognizance the additional gravitational potential from the disc. In their work [12] have included the gravitational potential from the disc in the study of the CR3BP. Also, [5] examined the combined effect of radiation and oblateness of both primaries, together with additional gravitational potential from the disc on the motion of an infinitesimal body in the CR3BP. They [5], studied the effects of oblateness up to J4 of the less massive primary and gravitational potential from the disc on the linear stability of triangular equilibrium points in the photogravitational CR3BP, and found that the gravitational potential from the disc makes the out-look of the CR3BP quite different such that new libration points exist. Hence this paper examines non-collinear equilibrium points of a circumbinary planet in orbit around luminous primaries moving in elliptic orbits about their common barycenter, surrounded by circular disc type. Section 1 introduces the research problem, while sections 2 and 3 presents the equations of motion and locates the non-collinear points, section 4 establishes the nature of stability of the equilibrium points, finally, sections 5 and 6 presents the numerical application to TV Crateris (HD 98800 B) and conclusions respectively.

2. Equations of Motion

The equations of motion of an infinitesimal circumbinary planet in the ER3BP with two finite bodies which are sources of radiation in a dimensionless-pulsating coordinate system (ξ, η, ζ) are given as in [10] and an additional potential from the disc as in [5] as:

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Omega_{\xi} \\ \ddot{\eta} + 2\dot{\xi} &= \Omega_{\eta} \\ \ddot{\zeta} &= \Omega_{\zeta} \end{aligned} \tag{1}$$

Where the force function is

$$\Omega = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{\xi^2+\eta^2}{2} + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{M_b}{(r^2+T^2)^{\frac{1}{2}}}\right\}\right] \tag{2}$$

$$\begin{aligned} r_1^2 &= (\xi+\mu)^2 + \eta^2 + \zeta^2 \\ r_2^2 &= (\xi+\mu-1)^2 + \eta^2 + \zeta^2 \end{aligned} \tag{3}$$

And the mean motion is

$$n^2 = \frac{1}{a}\left(1 + \frac{3}{2}e^2 + \frac{2M_b r_c}{(r^2+T^2)^{\frac{3}{2}}}\right) \tag{4}$$

Where n, a, e, are the mean motion, semi-major axis, eccentricity of the orbits of the primaries respectively; $\frac{M_b}{(r^2+T^2)^{\frac{1}{2}}}$ is the potential due to the circular accumulation of material points [13], r is the radial distance of the infinitesimal body and is given by $r^2 = \xi^2 + \eta^2$, $T = b + d$, b and d

are parameters determining the density profile of the circular accumulation of material points. The parameter b controls the flatness of the profile and is known as the flatness parameter. The parameter d controls the size of the core of the density profile and is called the core parameter. When $b = d = 0$, the potential becomes the same as that of a point mass, r_c is the radial distance of the infinitesimal body in the classical restricted 3BP (Singh and Taura, 2013) and is given as $r_c = 1 - \mu + \mu^2$

3. Locations of triangular equilibrium points

These are the points at which the velocity and acceleration of an infinitesimal mass is zero i.e $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$

Differentiating the force function Ω with respect to $\xi, \eta,$ *and* ζ and equating to zero this give

$$\Omega_\xi = \frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)q_1}{r_1^3} + \frac{\mu(\xi+\mu-1)q_2}{r_2^3} + \frac{M_b\xi}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right\}\right] = 0 \tag{5}$$

Since $\Omega_\xi = 0$

$$\Rightarrow n^2\xi - \frac{(1-\mu)(\xi+\mu)q_1}{r_1^3} - \frac{\mu(\xi+\mu-1)q_2}{r_2^3} - \frac{M_b\xi}{\left(r^2+T^2\right)^{\frac{3}{2}}} = 0 \tag{6}$$

Also

$$\Omega_\eta = \frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\eta - \frac{1}{n^2}\left\{\frac{(1-\mu)\eta q_1}{r_1^3} + \frac{\mu\eta q_2}{r_2^3} + \frac{M_b\eta}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right\}\right] = 0 \tag{7}$$

Since $\Omega_\eta = 0$

$$\Rightarrow n^2\eta - \frac{(1-\mu)\eta q_1}{r_1^3} - \frac{\mu\eta q_2}{r_2^3} - \frac{M_b\eta}{\left(r^2+T^2\right)^{\frac{3}{2}}} = 0 \tag{8}$$

Also

$$\Omega_\zeta = -\frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[\frac{1}{n^2}\left\{\frac{(1-\mu)\zeta q_1}{r_1^3} + \frac{\mu\zeta q_2}{r_2^3}\right\}\right] = 0 \tag{9}$$

Since $\Omega_\zeta = 0$

$$\Rightarrow -\zeta\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu q_2}{r_2^3}\right\} \tag{10}$$

From the above equation when $\zeta = 0$

$$\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu q_2}{r_2^3} \neq 0 \tag{11}$$

This indicates that the equilibrium points exist in the $\xi\eta$–plane in which the motion of the infinite masses occur i.e the motion of the infinitesimal mass occurs in the orbital plane. Hence from equation (8) $\eta = 0$ or

$$\Rightarrow n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{M_b}{\left(r^2 + T^2\right)^{\frac{3}{2}}} = 0 \tag{12}$$

The triangular points denoted by $L_{4,5}$ are the solutions of equations (6) and (8) when $\eta \neq 0$

From equation (6) by expanding and collecting terms in ξ this give

$$\xi\left[n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{M_b}{\left(r^2 + T^2\right)^{\frac{3}{2}}}\right] - \frac{\mu(1-\mu)q_1}{r_1^3} + \frac{\mu(1-\mu)q_2}{r_2^3} = 0 \tag{13}$$

Subsitituting equation (12) in equation (13),

$$\Rightarrow -\frac{\mu(1-\mu)q_1}{r_1^3} + \frac{\mu(1-\mu)q_2}{r_2^3} = 0 \tag{14}$$

$$\Rightarrow \mu(1-\mu)\left\{\frac{q_1}{r_1^3} - \frac{q_2}{r_2^3}\right\} = 0$$

This indicate either $\mu(1-\mu) = 0$ or $\left\{\frac{q_1}{r_1^3} - \frac{q_2}{r_2^3}\right\} = 0$

Now using $\mu(1-\mu) = 0$, we obtain $\mu = 0$ or $\mu = 1$

Substituting these values into equation (12) with $\mu = 0$

$$n^2 - \frac{q_1}{r_1^3} - \frac{M_b}{\left(r^2 + T^2\right)^{\frac{3}{2}}} = 0$$

(15)

For $\mu = 1$

$$n^2 - \frac{q_2}{r_2^3} - \frac{M_b}{\left(r^2 + T^2\right)^{\frac{3}{2}}} = 0$$

(16)

In the absence of potential from the circular cluster of disc, equations (15) and (16) becomes

$$n^2 - \frac{q_1}{r_1^3} = 0 \quad and \quad n^2 - \frac{q_2}{r_2^3} = 0$$

(17)

By the use of equation (17)

$$r_1 = \left(\frac{q_1}{n^2}\right)^{\frac{1}{3}}$$

$$r_2 = \left(\frac{q_2}{n^2}\right)^{\frac{1}{3}}$$

(18)

Substituting the mean motion and considering the radiation of the primaries, equation (18) will change slightly by $\varepsilon_i\,(i = 1,2)$, where $\varepsilon_i << 1$ and depends upon radiation and potential from the disc, equation (18) becomes

$$r_1 = (aq_1)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}}\right) + \varepsilon_1 \tag{19}$$

$$r_2 = (aq_2)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}}\right) + \varepsilon_2 \tag{20}$$

Where ε_1 *and* ε_2 are small quantities, now from equation (15), substituting r_1 *and* n^2 and neglecting products of e, M_b, ε_1 *and* ε_2

$$\varepsilon_1 = \frac{M_b a^{\frac{4}{3}} q_1^{\frac{1}{3}}}{3\left(r^2 + T^2\right)^{\frac{3}{2}}} \tag{21}$$

Similarly substituting r_2 *and* n^2 into equation (16)

$$\varepsilon_2 = \frac{M_b a^{\frac{4}{3}} q_2^{\frac{1}{3}}}{3\left(r^2 + T^2\right)^{\frac{3}{2}}} \tag{22}$$

Putting equation (21) and (22) into equation (19) and (20) results in

$$r_1 = (aq_1)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} + \frac{M_b a}{3\left(r^2 + T^2\right)^{\frac{3}{2}}}\right) \tag{23}$$

$$r_2 = (aq_2)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} + \frac{M_b a}{3\left(r^2 + T^2\right)^{\frac{3}{2}}}\right) \tag{24}$$

Now since $r^2 = \xi^2 + \eta^2 \Rightarrow r_c{}^2 = 1 - \mu + \mu^2$ equation (23) and (24) can be written as

$$r_1{}^2 = (aq_1)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}}\right) \tag{25}$$

$$r_2{}^2 = (aq_2)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}}\right) \tag{26}$$

Substituting the values of equation (25) and (26) into

$$\xi = \frac{1}{2} - \mu + \frac{r_1^2 + r_2^2}{2} \quad and \quad \eta^2 = r_1^2 - (\xi^2 + \mu)^2 \tag{27}$$

This gives

$$\xi = \frac{1}{2} - \mu + \frac{1}{2}\left[(aq_1)^{\frac{2}{3}} \left(1 - e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} \right) - (aq_2)^{\frac{2}{3}} \left(1 - e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} \right) \right] \tag{28}$$

$$\eta = \pm \left[(aq_1)^{\frac{2}{3}} \left(1 - e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} \right) \right.$$

$$\left. - \frac{1}{4}\left[1 + 2(aq_1)^{\frac{2}{3}} \left(1 - e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} \right) - 2(aq_2)^{\frac{2}{3}} \left(1 - e^2 - \frac{2M_b(2r_c - a)}{3\left(r_c^2 + T^2\right)^{\frac{3}{2}}} \right) \right] \right]^{\frac{1}{2}} \tag{29}$$

Equations (28) and (29) are the locations of the non-collinear points corresponding to $L_{4,5}$.

4. Stability of triangular equilibrium points

Denoting the positions of equilibrium points by (ξ_0, η_0) and small displacements α and β , Such that $\xi = \xi_0 + \alpha, \; and \; \eta = \eta_0 + \beta$ substituting these values into equation (1) we obtain the variational equations

$$\ddot{\alpha} - 2\dot{\beta} = \alpha\Omega_{\xi\xi}^0 + \beta\Omega_{\xi\eta}^o$$

$$\ddot{\beta} + 2\dot{\alpha} = \alpha\Omega_{\eta\xi}^0 + \beta\Omega_{\eta\eta}^o$$

Leading to the derivation of the characteristic equation

$$\lambda^4 - (\Omega_{\xi\xi}^0 + \Omega_{\eta\eta}^0 - 4)\lambda^2 + \Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 - (\Omega_{\xi\eta}^0)^2 = 0 \tag{30}$$

Now, evaluating the partial derivatives of the triangular points at equilibrium gives

$$\begin{aligned}
\Omega^0_{\xi\xi} = \left(1-e^2\right)^{-\frac{1}{2}} \Bigg[& \frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} + \frac{3\mu}{4(aq_2)^{\frac{2}{3}}} - \frac{3\mu q_1^{\frac{2}{3}}}{2q_2^{\frac{2}{3}}} + \frac{3\mu}{2} + e^2\left\{\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{3\mu}{4(aq_2)^{\frac{2}{3}}}\right\} \\
& - \frac{3M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} - \frac{\mu M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a q_1^{-\frac{2}{3}} q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{3\mu M_b a q_1^{-\frac{2}{3}} q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} \\
& + \frac{3\mu M_b a q_1^{\frac{2}{3}} q_2^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{\mu M_b(4r_c-5a)(aq_2)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a}{(r_c^2+T^2)^{\frac{5}{2}}}\Bigg\{\frac{1}{4} - \mu + \mu^2 + \frac{1}{2}(aq_1)^{\frac{2}{3}} - \frac{1}{2}(aq_2)^{\frac{2}{3}} \\
& - \mu(aq_1)^{\frac{2}{3}} + \mu(aq_2)^{\frac{2}{3}} + \frac{1}{4}(aq_1)^{\frac{4}{3}} + \frac{1}{4}(aq_2)^{\frac{4}{3}} - \frac{1}{2}a^{\frac{4}{3}} q_1^{\frac{2}{3}} q_2^{\frac{2}{3}}\Bigg\}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\Omega^0_{\eta\eta} = \left(1-e^2\right)^{-\frac{1}{2}} \Bigg[& -\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} - \frac{3\mu}{4(aq_2)^{\frac{2}{3}}} + \frac{3\mu q_1^{\frac{2}{3}}}{2q_2^{\frac{2}{3}}} + \frac{3\mu}{2} + e^2\left\{-\frac{3(1-\mu)}{4(aq_1)^{\frac{2}{3}}} - \frac{3\mu}{4(aq_2)^{\frac{2}{3}}}\right\} \\
& - \frac{3M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} + \frac{\mu M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} - \frac{3M_b a q_1^{-\frac{2}{3}} q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3\mu M_b a q_1^{-\frac{2}{3}} q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} \\
& - \frac{3\mu M_b a q_1^{\frac{2}{3}} q_2^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{\mu M_b(4r_c-5a)(aq_2)^{-\frac{2}{3}}}{4(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a}{(r_c^2+T^2)^{\frac{5}{2}}}\Bigg\{\frac{1}{2}(aq_1)^{\frac{2}{3}} + \frac{1}{2}(aq_2)^{\frac{2}{3}} - \frac{1}{4} \\
& - \frac{1}{4}(aq_1)^{\frac{4}{3}} - \frac{1}{4}(aq_2)^{\frac{4}{3}} + \frac{1}{2}a^{\frac{4}{3}} q_1^{\frac{2}{3}} q_2^{\frac{2}{3}}\Bigg\}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\Omega^0_{\xi\eta} = \left(1-e^2\right)^{-\frac{1}{2}} \Bigg[& \frac{3(1-2\mu)}{2} + \frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}} - \frac{3(1-\mu)q_2^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} - \frac{3\mu}{2(aq_2)^{\frac{2}{3}}} + \frac{3\mu q_1^{\frac{2}{3}}}{2q_2^{\frac{2}{3}}} + e^2\left\{\frac{3(1-\mu)}{2(aq_1)^{\frac{2}{3}}} - \frac{3\mu}{2(aq_2)^{\frac{2}{3}}}\right\} \\
& - \frac{3M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{3\mu M_b a}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{\mu M_b(4r_c-5a)(aq_1)^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a q_1^{-\frac{2}{3}} q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} \\
& - \frac{3\mu M_b a q_1^{-\frac{2}{3}} q_2^{\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{3\mu M_b a q_1^{\frac{2}{3}} q_2^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} - \frac{\mu M_b(4r_c-5a)(aq_2)^{-\frac{2}{3}}}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{3M_b a}{(r_c^2+T^2)^{\frac{5}{2}}}\left\{\frac{1}{2} - \mu + \frac{1}{2}(aq_1)^{\frac{2}{3}} - \frac{1}{2}(aq_2)^{\frac{2}{3}}\right\}\Bigg]
\end{aligned}$$

(31)

Substituting the above equations (31) into the characteristic equation and restricting ourselves to only linear terms in $e2, a, q_1, q_2$ and Mb where for simplicity let $a = 1 - \alpha$ and $q_i = 1 - \beta i$, where $i = 1,2$,

$$4(\lambda^2)^2 + 4(4 - 3\psi_1)\lambda^2 + 27\mu(1-\mu) + 4\psi_2 = 0 \tag{32}$$

Where

$$\psi_1 = (1-e^2)^{-\frac{1}{2}}\left(1 - \frac{M_b}{(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2 + T^2)^{\frac{5}{2}}}\right)$$

And

$$\psi_2 = 3\mu(1-\mu)\alpha + \frac{45\mu(1-\mu)}{4}e^2 + \frac{3\mu(1-\mu)}{2}(\beta_1 + \beta_2) + \frac{3\mu M_b(1-\mu)(4r_c - 11)}{2(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{27\mu M_b(1-\mu)}{4(r_c^2 + T^2)^{\frac{5}{2}}}$$

Equation (32) is a quadratic equation in λ2 which gives

$$\lambda^2 = \frac{-4(4 - 3\psi_1) \pm \left[(4 - 3\psi_1)^2 - 27\mu(1-\mu) - 4\psi_2\right]^{\frac{1}{2}}}{2}$$

λ is defined to be purely imaginary for stable motion, this implies that, the motion of the infinitesimal must be bounded and periodic, such that λ2 < 0. This implies that 3ψ1 - 4 ≤ 0
and the discriminant

$$\Delta = (4 - 3\psi_1)^2 - 27\mu(1-\mu) - 4\psi_2 \tag{33}$$

This yield

$$0 < e \leq \left[1 - \frac{9}{16}\left(1 - \frac{M_b}{(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2 + T^2)^{\frac{5}{2}}}\right)^2\right]^{\frac{1}{2}} \tag{34}$$

When $M_b = 0$ equation (34) becomes

$$0 < e \leq \frac{\sqrt{7}}{4} \tag{35}$$

If ψ1 and ψ2 are not satisfied the characteristics roots will be either real or complex conjugate. The positive real part indicates instability of the investigated triangular points in case of complex roots. Now, by the use of equations ψ1 and ψ2 in equation (33), this gives

$$
\begin{aligned}
\Delta = & \left(27+12\alpha+45e^2+6\beta_1+6\beta_2+\frac{6M_b(4r_c-11)}{(r_c^2+T^2)^{\frac{3}{2}}}+\frac{27M_b}{(r_c^2+T^2)^{\frac{5}{2}}}\right)\mu^2 \\
& -\left(27+12\alpha+45e^2+6\beta_1+6\beta_2+\frac{6M_b(4r_c-11)}{(r_c^2+T^2)^{\frac{3}{2}}}+\frac{27M_b}{(r_c^2+T^2)^{\frac{5}{2}}}\right)\mu \\
& +1-3e^2+\frac{6M_b}{(r_c^2+T^2)^{\frac{3}{2}}}-\frac{6M_b r_c^2}{(r_c^2+T^2)^{\frac{5}{2}}}
\end{aligned}
\qquad (36)
$$

If Δ= 0, then

$$
\mu_c=\frac{1}{2}\left(1-\frac{\sqrt{69}}{9}\right)-\frac{4}{27\sqrt{69}}\alpha-\frac{14}{9\sqrt{69}}e^2-\frac{2}{27\sqrt{69}}\left(\beta_1+\beta_2\right)+\left[\frac{(76-8r_c)(r_c^2+T^2)^{\frac{5}{2}}-9(1+6r_c^2)(r_c^2+T^2)^{\frac{3}{2}}}{27\sqrt{69}(r_c^2+T^2)^4}\right]M_b
$$

(37)

The critical value μc is the solution of the quadratic equation Δ = 0. It represents the effects of the various parameters involved on the size of the region of stability. A close examination shows that they all cause a reduction in the size of the region confirming their destabilizing tendencies, while the potential from the circumbinary disc has a stabilizing effect.

5. Numerical Computations of Triangular Equilibrium Points

The triangular points are obtained numerically for the binary system HD 98800B. This is a binary system with an elliptic orbit, radiating primaries surrounded by a disc. The table below contains numerical data about the binary system.

Table 1. Relevant numerical data

Binary system	Mass (M_o)		Luminosity(L⊙)		Eccentricity(e)
	m_1	m_2	L_1	L_2	e
HD 98800B	0.70	0.58	0.33	0.17	0.5

On the basis of Stefan-Boltzmann's law and considering K=1, the radiation pressure coefficients q_1 and q_2 are calculated, where $q_i = 1-\frac{A_iK_iL_i}{r_i\rho_iM_i}, (i = 1,2)$, where M is the mass and L the luminosity of a star respectively; r is the radius and ρ the density of a moving body; K is the radiation pressure efficiency factor of a star; $A=\frac{3}{16\pi CG}$ is a constant given by $A = 2.9838 \times 10^{-5}$ in the CGS system.

In order to show the effects of the various parameters on the positions of triangular points $L_{4,5}$ we consider the following cases using equations (28) and (29) which is shown in table 2.

1. Absence of eccentricity, semi-major axis, radiation, and potential from the belt (classical case).
2. Radiation of the bigger primary only
3. Radiation of the smaller primary only,
4. Eccentricity only
5. Semi-major axis only

6. Potential from the disc only.
7. Presence of eccentricity, semi-major axis, radiation of the primaries, and potential from the disc

The circular case *($e = M_b = 0$, and $a = q_1 = q_2 = 1$)* is first shown and *$T=0.01$* for simplicity.

Table 2. Location of triangular equilibrium points

Binary system	Mass ratio	Radiation parameters		Potentials			Locations of Δ points	
	(μ)	q_1	q_2	e	a	M_b	ξ	$\pm\eta$
HD 98800B	0.453125	1	1	0	1	0	0.0468750	0.866025
		0.99903	1	0	1	0	0.0465516	0.865839
		1	0.99941	0	1	0	0.0470717	0.865912
		1	1	0.6	1	0	0.0468750	0.624500
		1	1	0	0.65	0	0.0468750	0.707368
		1	1	0	1	0.01	0.0468750	0.861681
		0.99903	0.99941	0.6	0.65	0.01	0.0468152	0.470825

The effect of the various individual parameters on $L_{4,5}$ will be considered together with their corresponding diagrams using equations (28) and (29)

Table 3.Effect of eccentricity on $L_{4,5}$ of the binary system HD 98800 B

e	ξ	±η
0.7	0.0468113	0.502228
0.5	0.0467809	0.701505
0.3	0.0467607	0.807481
0.2	0.0467543	0.837854
0.1	0.0467505	0.855560

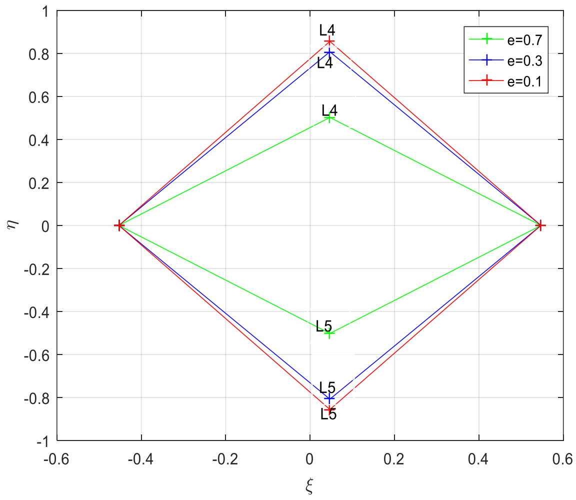

Fig 1: Effect of eccentricity on $L_{4,5}$ of the binary system HD 98800B

Table 4.Effect of Semi-major axis on $L_{4,5}$ of the binary system HD 98800B

a	ξ	$\pm\eta$
0.95	0.0467976	0.600350
0.85	0.0468033	0.562031
0.75	0.0468091	0.519359
0.65	0.0468152	0.470825
0.55	0.0468216	0.413860

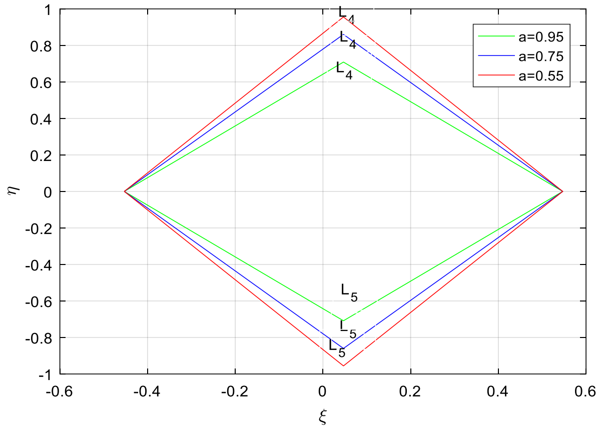

Fig 2: Effect of semi-major axis on $L_{4,5}$ in the binary system HD 98800B

Table 5.Effect of potential from the disc on $L_{4,5}$ in the binary system HD 98800B

M_b	ξ	$\pm\eta$
0.01	0.0468152	0.470825
0.03	0.0468173	0.452830
0.05	0.0468194	0.434089
0.1	0.0468247	0.383249
0.2	0.0468352	0.252533

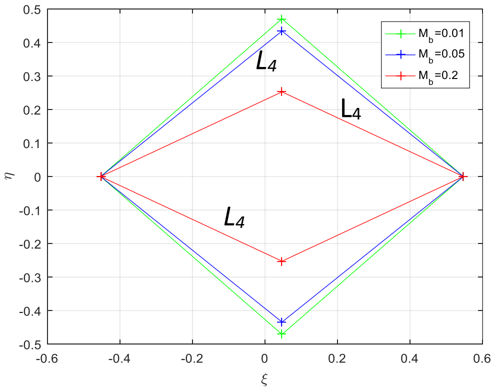

Fig 3: Effect of potential from the disc on $L_{4,5}$ in the binary system HD 98800B

L_5

Table 6: Effect of oblateness of the third body on $L_{4,5}$ in the binary system HD98800B

A	ξ	$\pm\eta$
0	0.0467525	0.846217
0.01	0.0467546	0.852500
0.02	0.0467566	0.858737
0.03	0.0467582	0.866939
0.1	0.0467728	0.907089
0.2	0.0467931	0.964125

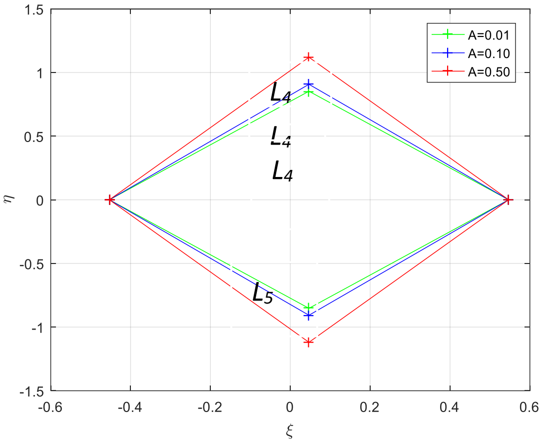

Figure 4: Effect of oblateness of the third body on $L_{4,5}$ in the binary system HD98800 B

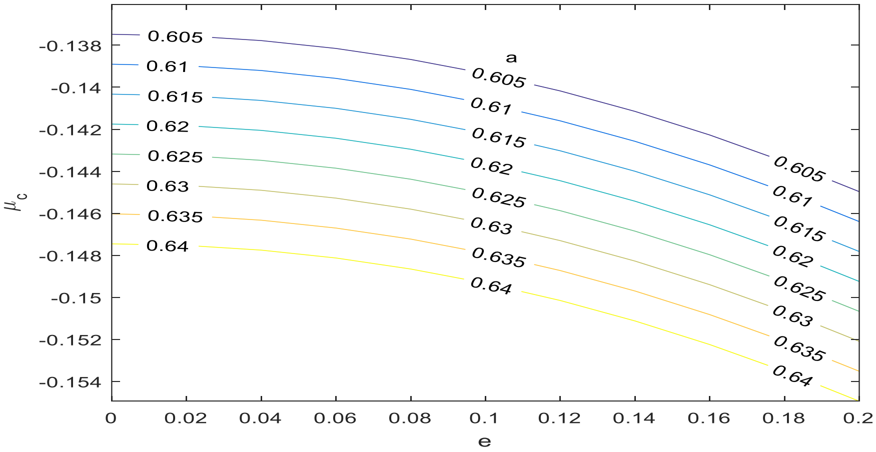

Fig 5: Effect of semi-major axis on the stability region in HD 98800B

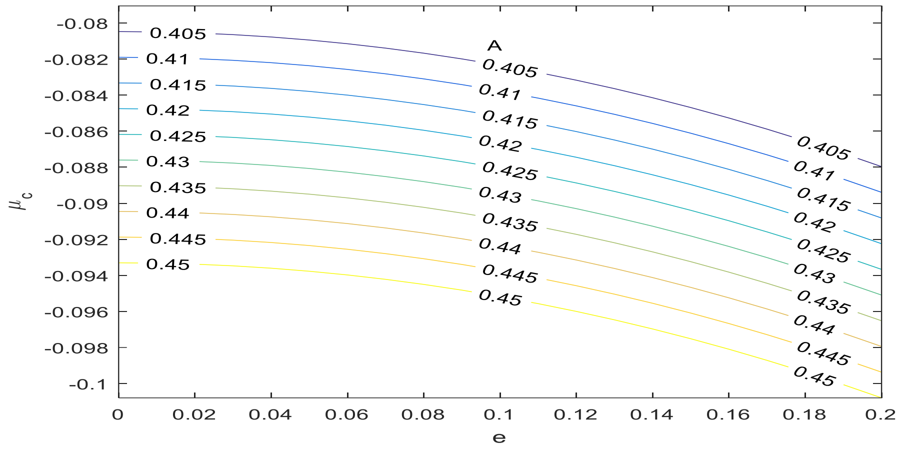

Fig 6: Effect of oblateness on the stability region in HD 98800B

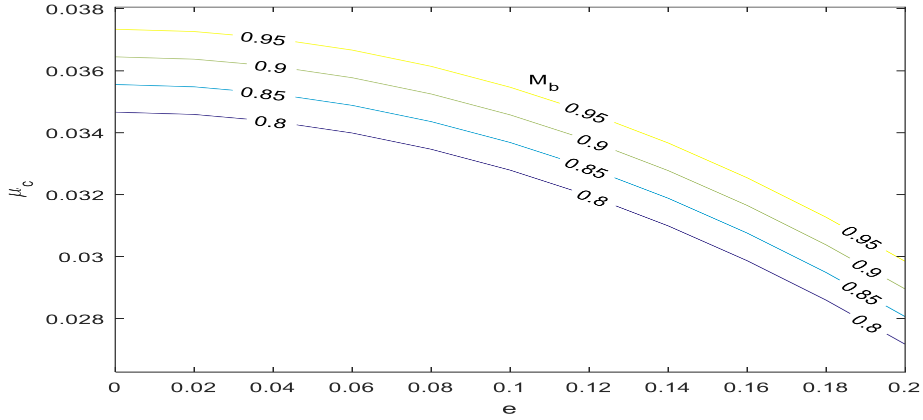

Fig 7: Effect of potential from the disc on the stability region in HD 98800 B

6. Discussion

Following the equation of motions [10], and [5], the non-collinear equilibrium points have been found and are shown analytically, numerically and graphically to depend on the radiation pressure factors of both primaries, eccentricity of the orbits and the gravitational potential from a circumbinary disc. The locations and nature of stability of the triangular equilibrium points reveals the effects of the various parameters on the dynamics of the system. The locations of these triangular equilibrium points are significantly dependent on the parameters. The stability of the triangular equilibrium points coincides with [8] when eccentricity, semi-major axis, radiation, and gravitational potential from a disc are relaxed, and is conditionally stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le ½$. It is observed that for increase in eccentricity and gravitational potential from a disc there is a shift of $L_{4,5}$ towards the line joining the primaries and away from the smaller primary (see tables 3, and 5 and figures 1, and 3). The increase in oblateness and semi-major axis cause $L_{4,5}$ to move away from the line joining the primaries and there is a slight shift of $L_{4,5}$ towards the bigger primary (see table 4, and 6 and figures 2 and 6).

In the case of stability, the effects of eccentricity, semi-major axis, and radiation all reduces the size of the region of stability, while the gravitational potential from a disc increases the size of the region of stability (see figures 5, 6 and 7). This confirms the results of [5] and [10]. The gravitational potential from the disc is the only stabilizing force.

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