



Bayesian Estimation of a Shape Parameter of a Weibull-Lomax Distribution Using Non-Informative Priors

^{*1}Mohammed Abubakar Usman and ²Baba Bukar Alhaji

^{*1}Department of Computer Science, School of General and Applied Sciences Shehu Idris College of Health Sciences and Technology, Makarfi Kaduna.

²Department of Mathematical Sciences, Nigerian Defence Academy, Kaduna

Corresponding Author: muhammadabuothman@gmail.com

Abstract

The Weibull-Lomax Distribution has several applications in the areas of survival analysis, especially where survival chances of an event decrease with time. It has positive skewness and has been found to perform better than most family of Lomax distributions such as Lomax, Gamma-Lomax, Beta-Lomax, exponentiated-Lomax distributions. In this study, the shape parameter of the Weibull-Lomax Distribution is estimated using the Bayesian approach under two non-informative prior distributions. The non-informative priors considered are the Uniform and Jeffrey's prior distributions. The squared error loss function (SELF), Quadratic loss function (QLF) and precautionary loss function (PLF) are used to derive the Bayesian estimators under each prior. The posterior distribution of the shape parameter of the Weibull-Lomax distribution. Furthermore, using simulation studies, the estimates are compared with the classical method (such as maximum likelihood) by employing biases and Mean Square Errors (MSEs). The result shows the Bayesian approach using Quadratic Loss Function (QLF) under both Uniform and Jeffrey's priors produced the best estimates for the shape parameter.

Keywords: Weibull Distribution, Lomax Distribution, Bayesian Maximum likelihood, Shape parameter.

1. Introduction

Estimation of parameters can be viewed in two main philosophical approaches. The classical and the Bayesian approaches. In classical or frequentist approach, the parameters are considered to be fixed, where as in the Bayesian paradigm, the parameters are view as unknown random variables. However, in many real life situations represented by lifetime models, the parameters cannot be treated as constant throughout the life-testing period and hence the need for Bayesian estimation for life time models. Bayesian inference has found application in a wide range of activities, including Science, Engineering, Philosophy, Medicine, Sport and Law. In the philosophy of decision theory, Bayesian inference is closely related to subjective probability, often called 'Bayesian probability' in which we combine new information that is available with the prior information we have, to form the basis for the statistical procedures. A prior distribution is said to non-informative if we have a little or no information a priori, about the distribution of the parameter we wish to estimate. This has little impact on the posterior and allows the data to speak for themselves. The non-informative priors considered in the research are the uniform and Jeffrey's' priors. Tahir et al. (2015) introduced a new model called the Weibull-Lomax distribution that extends the Lomax distribution. Various statistical properties of the new distribution were derived. The model parameters were estimated by the method of maximum likelihood and the observed information matrix was determined. The potentiality of

the new model was illustrated by means of two real life data sets. For these data, the new model outperformed the McDonald-Lomax, Kumaraswamy-Lomax, gamma-Lomax, beta-Lomax, exponentiated Lomax and Lomax models (see Tahir et al. (2015)). Aliyu and Yahaya. (2016) studied the scale parameter of generalized Rayleigh distribution under non-informative prior. Extended Jeffrey's prior with three loss functions namely; Entropy Loss Function, Precautionary Loss Function, Squared Error Loss Function were used. The method of maximum likelihood (MLE) was also employed Extensive Monte Carlo simulations was carried out to compare the performances of the Bayes estimates with that of MLEs. It was observed that the estimate under the Entropy loss function is more stable than the estimates under squared error loss function, Precautionary loss function and MLEs. Ahmad et al. (2016) in their study compared the classical and non-classical methods of estimation for a Nakagami distribution. The frequentist method of maximum likelihood estimator was obtained. The Bayesian estimation method was also used in order to estimate the scale parameter of Nakagami distribution by using Jeffreys', Extension of Jeffreys', and Quasi priors under three different loss functions with a simulation study conducted in R software. Ieren and Chukwu (2018) employed three different loss functions in the Bayesian estimation of a Shape Parameter of the Weibull-Frechet Distribution. For other Bayesian estimation of various distributions, see the following; Himanshu and Arun (2009), Dey and Maiti (2010), Preda et al. (2010), Sanku (2010), Sanku and Tanujit (2011), Sajid et al. (2012), Navid and Muhammad (2012), Feroze (2012), Almutairi and Heng (2012), Dey et al. (2013), Azam and Ahmad (2014), Wasif and Navid (2015), Wajiha and Muhammed (2015), Gupta and Singh (2017), Gupta (2017), Ieren and Oguntunde (2018), Mabur et al (2018), Eraikhuemen et al. (2020a), Eraikhuemen (2020), among others.

1.1 Motivation

When using Bayesian method of estimation, we combine any new information that is available with the prior information to form the basis for statistical inference. Tahir et al. (2015) developed and studied an extension of the Lomax distribution called Weibull-Lomax distribution with four parameters. Many important properties of this distribution were extensively derived and discussed with applications to real life data sets. Tahir et al. (2015) demonstrated that the Weibull-Lomax distribution is a flexible model and performed better than some existing forms of the Lomax distribution such as McDonald-Lomax, Kumaraswamy-Lomax, Gamma-Lomax, Beta-Lomax, Exponentiated-Lomax and Lomax Distributions. The model has wider application in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others. The flexibility and performance of the model has motivated us to estimate the shape parameter of the Weibull-Lomax distribution proposed by Tahir et al. (2015) using Bayesian approach and compare the results with the classical approach of estimation of Tahir et al. (2015). This paper is organized as follows: In Section 2, the Weibull Lomax Distribution is defined and the Classical and the Bayesian approaches to estimations of the distribution are derived. In Section 3, we discussed the simulation set up for the study. Section 4 presents analysis and results of the study, and finally we conclude the study in Section 5 with future research and contribution of the study.

2 Materials and Methods

2.1 The Weibull-Lomax Distribution

According Tahir et al. (2015), the probability density function (pdf) and the cumulative distribution function (cdf) of the Weibull-Lomax distribution are respectively given by;

$$f(x) = \frac{\alpha \lambda \theta}{\beta} \left[1 + \left(\frac{x}{\beta}\right)\right]^{\lambda \theta - 1} \left(1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\theta}\right)^{\lambda - 1} e^{-\alpha \left(\left[1 + \left(\frac{x}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda}} \quad (1)$$

and

$$F(x) = 1 - e^{-\alpha \left(\left[1 + \left(\frac{x}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda}}; x > 0, \alpha, \beta, \theta, \lambda > 0 \quad (2)$$

Where β is the scale parameter, and α, θ and λ are the shape parameters of the Weibull-Lomax distribution. Our interest in this study is to estimate only the shape parameter α , as the other shape parameters share the same properties in the distribution.

2.2 Maximum Likelihood Estimation Method

Given as $x_1, x_2, \dots, x_n$ a random sample from a population X with probability density function $f(x)$, the likelihood function, $L(X/\alpha, \beta, \theta, \lambda)$ is defined to be the joint density of the random sample $x_1, x_2, \dots, x_n$. The likelihood function of (2) is given by:

$$L(X/\alpha, \beta, \theta, \lambda) = \left(\frac{\alpha \lambda \theta}{\beta}\right)^n \prod_{i=1}^n \left[1 + \left(\frac{x_i}{\beta}\right)\right]^{\lambda \theta - 1} \times \prod_{i=1}^n \left(1 - \left[1 + \left(\frac{x_i}{\beta}\right)\right]^{-\theta}\right)^{\lambda - 1} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda}} \quad (3)$$

Let the likelihood function with respect to α be, $L(X/\alpha)$, therefore

$$L(X/\alpha) \propto \alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda}}$$

Let the log-likelihood function with respect to α be $l = \log L(X/\alpha)$ therefore,

$$l = n \log \alpha - \alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda} \quad (4)$$

Differentiating (4) partially with respect to α , the parameter of interest, and solving for $\hat{\alpha}$ gives:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda} = 0, \quad \hat{\alpha} = \frac{n}{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^{\theta} - 1\right)^{\lambda}} \quad (5)$$

2.3 Posterior Distributions

Assuming that the values $x = (x_1, x_2, \dots, x_n)$ are obtained independently, the likelihood function is given by:

$$L(\alpha | x) = P((x_1, x_2, \dots, x_n | \alpha)) = \prod_{i=1}^n P(X_i/\alpha)$$

To obtain the posterior distribution $P(\alpha | x)$, the probability distribution of the parameters once the data has been observed, we apply Bayes' Theorem

$$p(\alpha | \underline{X}) = \frac{p(\alpha)L(\alpha|\underline{X})}{\int_0^\infty p(\alpha)L(\alpha|\underline{X})d\alpha} \quad (6)$$

Where $p(\alpha)$ and $L(\alpha | \underline{X})$ are the prior distribution and the likelihood function respectively.

2.4 Posterior Distribution of the Shape Parameter under the Assumption of Uniform Prior

The uniform prior (non-informative prior) relating to the shape parameter α is defined as

$$p(\alpha) \propto 1, 0 < \alpha < \infty \quad (7)$$

The posterior distribution under uniform prior can be derived from (6). The likelihood function of the WLD in (3) for the shape parameter yields:

$$L(\alpha | \underline{X}) \propto \alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda} \quad (8)$$

Now, let

$$K_1 \int_0^\infty p(\alpha) L(\alpha | \underline{X}) d\alpha \quad (9)$$

by substitution we have:

$$K_1 = \int_0^\infty \alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda} d\alpha \quad (10)$$

Evaluating the integral we obtain the following:

$$K_1 = \frac{\Gamma(n+1)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda \right)^{n+1}} \quad (11)$$

Substituting (7), (8) and (11) into (6) and simplifying, we obtain the posterior distribution under uniform prior as follows:

$$p(\alpha|\underline{X}) = \frac{\alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda}}{\frac{\Gamma(n+1)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda \right)^{n+1}}}$$

$$p(\alpha|\underline{X}) = \frac{\alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda}}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta \right] - 1 \right)^\lambda \right)^{(n+1)} \Gamma(n+1)} \quad (12)$$

2.5 Posterior Distribution of the Shape Parameter under the Assumption of Jeffery's Prior

As another non-informative prior, the posterior distribution under Jeffery's prior is obtained as follows:

$$p(\alpha) \propto \frac{1}{\alpha}; 0 < \alpha < \infty \quad (13)$$

The likelihood is given in (14) below,

$$L(\underline{X}|\alpha) \propto e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)} \quad (14)$$

To obtain the posterior distribution of the shape parameter α for a given data defined in (6) and (9) using Jeffery's prior in (4) is as follows:

Solving for K_2 we have;

$$K_2 = \int_0^\infty \alpha^{n-1} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)} d\alpha \quad (15)$$

Evaluating the integration in (14), we obtain the following:

$$K_2 = \frac{\Gamma(n)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^n} \quad (16)$$

Substituting for (13), (14), and (15) in equation (6) and simplifying, we obtain the posterior distribution under Jeffery's prior as follows:

$$p(\alpha|\underline{X}) = \frac{\alpha^{-1} \alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)}}{\frac{\Gamma(n)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^n}}$$

$$\alpha|\underline{X}) = \frac{\alpha^{n-1} \left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)}}{\Gamma(n)} \quad (17)$$

2.6 Bayesian Estimation under Uniform Prior Using Three Loss Functions

Here, we estimate the shape parameter of the WLD under three loss functions using the posterior distribution obtained from the uniform prior. A Loss function, $L(\alpha, \alpha_{SELF})$ is that which describes the losses incurred by making an estimate $\hat{\alpha}$ of the true value of the parameter is α .

2.6.1 Squared Error Loss Function (SELF)

The squared error loss function relating to the shape parameter is defined from Azam and Ahmad (2014) as:

$$L(\alpha, \alpha_{SELF}) = (\alpha - \alpha_{SELF})^2 \quad (18)$$

Where α_{SELF} is the estimator of the parameter α under SELF.

The derivation of Bayes' estimator using SELF under uniform prior is obtained as:

$$\alpha_{SELF} = E(\alpha) = E(\alpha | \underline{X})$$

$$E(\alpha | \underline{X}) = \int_0^\infty \alpha P(\alpha | \underline{X}) d\alpha \quad (19)$$

The posterior distribution with uniform prior in (12),

$$\begin{aligned} P(\alpha | \underline{X}) &= \frac{\alpha^n e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)}}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^{-(n+1)} \Gamma(n+1)} \\ &= \frac{\alpha^n \left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^{-(n+1)} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)}}{\Gamma(n+1)} \end{aligned}$$

Substituting for $P(\alpha | \underline{X})$ in (19) we have:

$$\begin{aligned} E(\alpha | \underline{X}) &= \int_0^\infty \alpha^n \frac{\alpha \left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^{-(n+1)} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)}}{\Gamma(n+1)} d\alpha \\ E(\alpha | \underline{X}) &= \frac{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right) \right)^{-(n+1)}}{\Gamma(n+1)} \int_0^\infty \alpha^{n+1} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)} d\alpha \quad (20) \end{aligned}$$

Solving out the integral in (20), we obtain the following:

$$\begin{aligned} \alpha_{SELF} = E(\alpha | \underline{X}) &= \frac{\Gamma(n+2)}{\Gamma(n+1) \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)} \\ \alpha_{SELF} = E(\alpha | \underline{X}) &= \frac{n+1}{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)} \quad (21) \end{aligned}$$

2.6.2 Quadratic Loss Function (QLF)

The quadratic loss function as defined from Azam and Ahmad (2014) as:

$$L(\alpha, \alpha_{QLF}) = \left(\frac{\alpha - \alpha_{QLF}}{\alpha} \right)^2 \quad (22)$$

Where α_{QLF} is the estimator of the parameter α under QLF.

$$\alpha_{QLF} = \frac{E(\alpha^{-1})}{E(\alpha^{-2})} = \frac{E(\alpha^{-1}|\underline{X})}{E(\alpha^{-2}|\underline{X})}$$

$$E(\alpha^{-1}|\underline{X}) = \int_0^\infty \alpha^{-1} P(\alpha|\underline{X}) d\alpha \quad (23)$$

The posterior distribution with uniform prior in (12) and substituting $P(\alpha | \underline{X})$ in equation (23), we have:

$$E(\alpha^{-1}|\underline{X}) = \int_0^\infty \alpha^{-1} \frac{\alpha^n \left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)^{n+1} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda} \right)}{\Gamma(n+1)} d\alpha$$

$$E(\alpha^{-1}|\underline{X}) = \frac{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)^{n+1} \right)}{\Gamma(n+1)} \int_0^\infty \alpha^{n-1} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda} d\alpha \quad (24)$$

solving the integral in (24); we obtain the following:

$$E(\alpha^{-1}|\underline{X}) = \frac{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)^\lambda \Gamma(n)}{\Gamma(n+1)} \quad (25)$$

Similarly,

$$E(\alpha^{-2}|\underline{X}) = \int_0^\infty \alpha^{-2} P(\alpha|\underline{X}) d\alpha \quad (26)$$

The posterior distribution with uniform prior in (12) and substituting $P(\alpha | \underline{X})$ in equation (26), we have:

$$E(\alpha^{-2}|\underline{X}) = \int_0^\infty \alpha^{-2} \frac{\alpha^n \left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)^{n+1} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda} \right)}{\Gamma(n+1)} d\alpha$$

$$E(\alpha^{-2}|\underline{X}) = \frac{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)^{n+1} \right)}{\Gamma(n+1)} \int_0^\infty \alpha^{n-2} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda} d\alpha \quad (27)$$

solving the integral in (12), we obtain the following:

$$E(\alpha^{-2}|\underline{X}) = \frac{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right)^\theta - 1 \right]^\lambda \right)^{n+1} \right)^2 \Gamma(n-1)}{\Gamma(n+1)} \quad (28)$$

But

$$\alpha_{QLF} = \frac{E(\alpha^{-1})}{E(\alpha^{-2})} = \frac{E(\alpha^{-1}|\underline{X})}{E(\alpha^{-2}|\underline{X})}$$

This implies that

$$\begin{aligned}
\alpha_{QLF} &= \frac{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \Gamma(n)}{\Gamma(n+1)} \div \frac{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \right)^2 \Gamma(n-1)}{\Gamma(n+1)} \\
&= \frac{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \Gamma(n)}{\Gamma(n+1)} \times \frac{\Gamma(n+1)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \right) \Gamma(n-1)} \\
&= \frac{\Gamma(n)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \right)^2 \Gamma(n-1)} \\
&= \frac{n-1}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \right)}
\end{aligned} \tag{29}$$

2.6.3 Precautionary Loss Function (PLF)

The precautionary loss function (PLF) introduced by Norstrom (1996) is an asymmetric loss function and is defined from Azam and Ahmad (2014) as:

$$L(\alpha_{PLF}, \alpha) = \frac{(\alpha_{PLF} - \alpha)^2}{\alpha} \tag{30}$$

Where α_{PLF} is the estimator of the parameter α under PLF. Similarly, the derivation of Bayes estimator using PLF under uniform prior is obtained as follows:

$$\begin{aligned}
\alpha_{PLF} &= \{E(\alpha^2)\}^{\frac{1}{2}} = \{E(\alpha^2|\underline{X})\}^{\frac{1}{2}} = \sqrt{E(\alpha^2|\underline{X})} \\
E(\alpha^2|\underline{X}) &= \int_0^\infty \alpha^2 P(\alpha|\underline{X}) d\alpha
\end{aligned} \tag{31}$$

The posterior distribution with uniform prior in (12) and substituting $p(\alpha | \underline{X})$ in equation (31), we have:

$$E(\alpha^2|\underline{X}) = \frac{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \right)^{n+1}}{\Gamma(n+1)} \int_0^\infty \alpha^{n+2} e^{-\alpha \sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda} d\alpha \tag{32}$$

Similarly, solving the integral in (32), we obtain the following:

$$E(\alpha^2|\underline{X}) = \frac{\Gamma(n+3)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta} \right) \right]^\theta - 1 \right)^\lambda \right)^2 \Gamma(n+1)}$$

$$\begin{aligned}\alpha_{PLF} = \{E(\alpha^2)\}^{\frac{1}{2}} &= \sqrt{\frac{\Gamma(n+3)}{\left(\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^\theta - 1 \right)^\lambda \right)^2 \Gamma(n+1)}} \\ &= \frac{[(n+2)(n+1)]^{\frac{1}{2}}}{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^\theta - 1 \right)^\lambda}\end{aligned}\quad (33)$$

2.7 Bayesian Estimation under Jeffery's Prior Using Three Loss Functions

2.7.1 Squared Error Loss Function (SELF)

The derivation of Bayes estimator using SELF under Jeffrey's prior follows from the Bayes estimator using SELF under uniform prior given in equations (18) and (19). Consider the posterior distribution under the Jeffrey's prior in (17). Substituting (17) into (19), evaluating the integral in the expression, we obtain the Bayes estimator using SELF under Jeffrey's prior as:

$$\alpha_{SELF} = E(\alpha | \underline{X}) = \frac{n}{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^\theta - 1 \right)^\lambda} \quad (34)$$

2.7.2 Quadratic Loss Function (QLF)

The derivation of Bayes estimator using QLF under Jeffrey's prior follows from the Bayes estimator using QLF under uniform prior given in equations (23). Consider the posterior distribution under the Jeffrey's prior in (17). Substituting (17) into (23), evaluating the integral in the expression, we obtain the Bayes estimator using QLF under Jeffrey's prior as:

$$\alpha_{QLF} = \frac{n-2}{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^\theta - 1 \right)^\lambda} \quad (35)$$

2.7.3 Precautionary Loss Function (PLF)

Similarly, the derivation of Bayes estimator using PLF under Jeffrey's prior follows from the Bayes estimator using PLF under uniform prior given in equations (31). Consider the posterior distribution under the Jeffrey's prior in (17). Substituting (17) into (31), evaluating the integral in the expression, we obtain the Bayes estimator using QLF under Jeffrey's prior as:

$$\alpha_{PLF} = \{E(\alpha^2)\}^{\frac{1}{2}} = \sqrt{\frac{\Gamma(n-2)}{\sum_{i=1}^n \left(\left[1 + \left(\frac{x_i}{\beta}\right)\right]^\theta - 1 \right)^\lambda \Gamma(n)}} \quad (36)$$

3. Simulation Study

In this study, we generate random numbers from the Weibull-Lomax distribution with different values of shape and scale parameters to investigate the performance of the estimators, priors and loss functions. The Maximum Likelihood estimates, Bayesian estimates were obtained under a reasonably high number of replications, different sample sizes were used to investigate the performance of the estimators in relation to their biases and mean squared errors as well as the

sample sizes. The performance of the two methods was evaluated using the following performance measures:

$$\text{Bias (Absolute)} = \frac{1}{N} \sum_{i=1}^n |\hat{\alpha}_i - \alpha| \quad (37)$$

and Mean Square Error,

$$MSE = \frac{1}{N} \sum_{i=1}^n (\hat{\alpha}_i - \alpha)^2$$

Where $\hat{\alpha}$ is the estimate of α (the shape parameter) from the i^{th} simulated data and N is the number of Monte Carlo samples. The criteria for selection of best fit is that the estimate with the lowest bias and MSE will be considered as the best.

4 Analysis and Results

We compute empirically, the estimators derived using simulated data from the Weibull-Lomax distribution to obtain numerical estimates of parameters which enable us to measure the loss attached to the value of the estimated parameter against the true value of the shape parameter. We use simulation to check the performance of the priors and the loss functions. For this purpose we use different values of shape parameter α by taking different sample sizes. We generate random sample of size $n = (10, 15, 65, 95)$ from WLD using different parameter values. The following tables present the results of our simulation study by listing the estimates of the shape parameter under the appropriate estimation methods such as the Maximum Likelihood Estimation (MLE), Squared Error Loss Function (SELF), Quadratic Loss Function (QLF), and Precautionary Loss Function (PLF) under both Uniform and Jeffery prior.

Table1: Estimates Biases and Mean Squared Errors based on different sample sizes where $\alpha = 1$, $\lambda = 1$, $\theta = 1.0$ and $\beta = 1.0$

Sample Sizes		Measures		MLE Uniform Prior		Jeffery's Prior	
		SELF	QLF	PLF	SELF	QLF	PLF
10	Estimate	1.1056	1.2161	0.9950	1.2702	1.1056	1.1595
	BIAS	0.1056	0.2161	0.0050	0.2702	0.1056	0.1595
	MSE	0.1634	0.2309	0.1234	0.2740	0.1634	0.1929
15	Estimate	1.0757	1.1474	1.004	1.1827	1.0757	1.1109
	BIAS	0.0757	0.1474	0.004	0.1827	0.0757	0.1109
	MSE	0.0976	0.1262	0.080	0.1444	0.0976	0.1103
65	Estimate	1.0138	1.0294	0.9982	1.0372	1.0138	1.0216
	BIAS	0.0138	0.0294	0.0018	0.0372	0.0138	0.0174
	MSE	0.0166	0.0178	0.0159	0.0185	0.0166	0.0171
95	Estimate	1.0123	1.0230	1.0017	1.0283	1.0123	1.0176
	BIAS	0.0123	0.0230	0.0017	0.0283	0.0123	0.0090
	MSE	0.0111	0.0117	0.0107	0.0121	0.0111	0.0114

From Table 1 we can see that the estimator using QLF is better than the other estimators under both Uniform and Jeffrey priors with smaller values of bias and MSE irrespective of the variation in the samples. Hence, the Bayesian estimation (using QLF under Uniform and Jeffrey prior) for this parameter is better than Method of Maximum Likelihood estimation (MLE) for the chosen parameter values irrespective of small, medium or large sample sizes.

Table 2: Estimates, Biases and Mean Squared Errors based on different sample sizes where $\alpha = 2.5$, $\lambda = 1.0$, $\theta = 0.5$ and $\beta = 0.5$

Sample Sizes Measures		MLE Uniform Prior			Jeffery's Prior			
		SELF	QLF	PLF	SELF	QLF	PLF	
10	Estimate	2.7639	3.0403	2.4875	3.1755	2.7639	2.2111	2.8988
	BIAS	0.2639	0.5403	0.0125	0.6755	0.2639	0.2889	0.3988
	MSE	1.0213	1.4434	0.7710	1.7125	1.0213	0.6925	1.2059
15	Estimate	2.6892	2.8684	2.5099	2.9567	2.6892	2.3306	2.7774
	BIAS	0.1892	0.3684	0.0099	0.4567	0.1892	0.1694	0.2774
	MSE	0.6097	0.7887	0.5000	0.9024	0.6097	0.4598	0.6891
65	Estimate	2.5345	2.5735	2.4955	2.5929	2.5345	2.4565	2.5540
	BIAS	0.0345	0.0735	0.0045	0.0929	0.0345	0.0435	0.0540
	MSE	0.1036	0.1109	0.0993	0.1158	0.1036	0.0981	0.1069
95	Estimate	2.5308	2.5575	2.5042	2.5708	2.5308	2.4776	2.5441
	BIAS	0.0308	0.0575	0.0042	0.0708	0.0308	0.0224	0.0441
	MSE	0.0692	0.0730	0.0669	0.0755	0.0692	0.0659	0.0710

Table 2 also gives a similar pattern of the result found in Table 1 with lower values of biases and MSE for the estimators using QLF under both Uniform and Jeffrey prior. This result indicates that QLF under Jeffrey prior produces the best estimates followed by QLF under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used.

Table3: Biases and Mean Squared Errors based on different sample sizes where $\alpha = 2.5$, $\lambda = 1.0$, $\theta = 0.5$ and $\beta = 5.0$

Sample Sizes	Measures	MLE Uniform Prior			Jeffery's Prior			
		SELF	QLF	PLF	SELF	QLF	PLF	
10	Estimate	2.7639	3.0403	2.4875	3.1755	2.7639	2.2111	2.8988
	BIAS	0.2639	0.5403	0.0125	0.6755	0.2639	0.2889	0.3988
	MSE	1.0213	1.4434	0.7710	1.7125	1.0213	0.6925	1.2059
15	Estimate	2.6892	2.8684	2.5099	2.9567	2.6892	2.3306	2.7774
	BIAS	0.1892	0.3684	0.0099	0.4567	0.1892	0.1694	0.2774

	MSE	0.6097	0.7887	0.5000	0.9024	0.6097	0.4598	0.6891
65	Estimate	2.5345	2.5735	2.4955	2.5929	2.5345	2.4565	2.5540
	BIAS	0.0345	0.0735	0.0045	0.0929	0.0345	0.0435	0.0540
	MSE	0.1036	0.1109	0.0993	0.1158	0.1036	0.0981	0.1069
95	Estimate	2.5308	2.5575	2.5042	2.5708	2.5308	2.4776	2.5441
	BIAS	0.0308	0.0575	0.0042	0.0708	0.0308	0.0224	0.0441
	MSE	0.0692	0.0730	0.0669	0.0755	0.0692	0.0659	0.0710

Again from Table 3 it is obvious that QLF (under Jeffrey prior) yielded the best estimate for the shape parameter despite the changes in the sample sizes. This performance is still true of QLF (under Uniform prior) despite the variations in sample sizes.

Table 4: Estimates, Biases and Mean Squared Errors based on different sample sizes where $\alpha = 2.5$, $\lambda = 1.0$, $\theta = 5.0$ and $\beta = 0.5$

Sample Sizes		Measures		MLE		Uniform Prior		Jeffery's Prior		
				SELF		QLF		PLF	SELF	QLF
10	Estimate	2.7639	3.0403	2.4875	3.1755	2.7639	2.2111	2.8988		
	BIAS	0.2639	0.5403	0.0125	0.6755	0.2639	0.2889	0.3988		
	MSE	1.0213	1.4434	0.7710	1.7125	1.0213	0.6925	1.2059		
15	Estimate	2.6892	2.8684	2.5099	2.9567	2.6892	2.3306	2.7774		
	BIAS	0.1892	0.3684	0.0099	0.4567	0.1892	0.1694	0.2774		
	MSE	0.6097	0.7887	0.5000	0.9024	0.6097	0.4598	0.6891		
65	Estimate	2.5345	2.5735	2.4955	2.5929	2.5345	2.4565	2.5540		
	BIAS	0.0345	0.0735	0.0045	0.0929	0.0345	0.0435	0.0540		
	MSE	0.1036	0.1109	0.0993	0.1158	0.1036	0.0981	0.1069		
95	Estimate	2.5308	2.5575	2.5042	2.5708	2.5308	2.4776	2.5441		
	BIAS	0.0308	0.0575	0.0042	0.0708	0.0308	0.0224	0.0441		
	MSE	0.0692	0.0730	0.0669	0.0755	0.0692	0.0659	0.0710		

Again from Table 4, it is revealed that QLF (under Jeffrey prior) has the best estimate for the shape parameter despite the changes in parameter values and the sample sizes.

Table5: Biases and Mean Squared Errors based on different sample sizes where $\alpha = 5.0$, $\lambda = 1.0$, $\theta = 0.5$ and $\beta = 0.5$

Sample Sizes	Measures MLE		Uniform Prior			Jeffery's Prior	
	SELF		QLF	PLF	SELF	QLF	PLF
10	Estimate	5.5278 6.0806	4.9750	6.3510	5.5278	4.4223	5.7976
	BIAS	0.5278 1.0806	0.0250	1.3510	0.5278	0.5777	0.7976
	MSE	4.0851 5.7736	3.0839	6.8498	4.0851	2.7700	4.8234
15	Estimate	5.3783 5.7369	5.0198	5.9134	5.3783	4.6612	5.5547
	BIAS	0.3783 0.7369	0.0198	0.9134	0.3783	0.3388	0.5547
	MSE	2.4388 3.1550	2.0002	3.6096	2.4388	1.8391	2.7564
65	Estimate	5.0691 5.1471	4.9911	5.1859	5.0691	4.9131	5.1079
	BIAS	0.0691 0.1471	0.0089	0.1859	0.0691	0.0869	0.1079
	MSE	0.4142 0.4438	0.3970	0.4631	0.4142	0.3922	0.4274
95	Estimate	5.0617 5.1149	5.0084	5.1415	5.0617	4.9551	5.0882
	BIAS	0.0617 0.1149	0.0084	0.1415	0.0617	0.0449	0.0882
	MSE	0.2770 0.2922	0.2675	0.3019	0.2770	0.2638	0.2838

The above Table 5 also shows that Jeffrey prior with QLF gives us better estimates for the shape parameter, based on all the results presented in the tables, we can conclude that Bayes estimates using Quadratic loss function (QLF) under Jeffrey prior as well as QLF using Uniform prior are associated with minimum biases and MSE when compared to those obtained using MLE, SELF and PLF under Jeffrey prior and Uniform prior irrespective of the parametric values as well as the allocated sample sizes of $n = 10, 15, 65$ and 95 .

5 Conclusion

In this work, we obtain Bayesian estimators of the shape parameter of Weibull-Lomax Distribution. Using Uniform and Jeffrey priors, we derived the Posterior distributions of this parameter. Using three loss functions under the two prior distributions we also derived Bayes estimators. The three loss functions taken up are Squared Error Loss Function (SELF), Quadratic Loss Function (QLF) and Precautionary Loss Function (PLF). The performance of these estimators is assessed on the basis of their relative Biases and mean square errors. Monte Carlo Simulations are used to compute the estimates and the results are compared. It is discovered that using the Quadratic Loss Function (under Jeffrey prior) produces the least measures of bias and MSE, followed by the Quadratic Loss Function (under Uniform prior) then the Squared Error Loss Function, MLE and lastly the PLF under both Uniform and Jeffrey priors irrespective of the parameter values and differences in sample size. We found that Bayesian Approach using Quadratic Loss Function (QLF) under both Uniform and Jeffrey priors produces the best estimates of the parameter of interest compared to estimates using Maximum Likelihood approach, Squared

Error Loss Function (SELF) and Precautionary Loss Function (PLF) under both Uniform and Jeffrey priors irrespective of the values of the parameters and the different sample sizes.

This research article, we are able to achieved the following:

1. We derived the posterior distributions of the shape parameter of WLD assuming Uniform and Jeffrey priors using SELF, QLF and PLF.
2. We obtained the Estimates, Biases and Mean Square Errors of the posterior distributions of the shape parameter mathematically and empirically using Monte Carlo simulation techniques.
3. Following the comparative analysis of this study, we were able to establish that the shape parameter of the WLD can best be obtained using the method of Bayesian estimation using Quadratic Loss Function under the assumption of Jeffrey and Uniform prior.

As a future research, the study can look at the following:

1. The estimation of the same parameter using some other priors and loss functions.
2. Compare some other classical methods besides MLE with Bayesian approach base on different prior distributions and loss functions on the same parameter of the same distribution.
3. Estimate the shape parameter(s) of the WLD using some appropriate methods.

References

- Ahmad, K., Ahmad, S. P. & Ahmed A. Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Journal of Probability and Statistics*, Volume 2016, Article ID 7581918, 8 pages <http://dx.doi.org/10.1155/2016/7581918>
- Aliyu, Y. and Yahaya, A. (2016). Bayesian estimation of the shape parameter of generalized Rayleigh distribution under non-informative prior. *International Journal of Advanced Statistics and Probability*, 4 (1):1-10.
- Almutairi, A. O. and Heng, C. L. (2012). Bayesian Estimate for Shape Parameter from Generalized Power Function Distribution. *Mathematical Theory and Modeling*, II (12).
- Azam, Z. and Ahmad, S. A. (2014). Bayesian Approach in Estimation of Scale Parameter of Nakagami Distribution. *Pakistan Journal of Statistics and Operations Research*, 10(2), 217-228.
- Dey, S., and Maiti, S. S. (2010). Bayesian estimation of the parameter of Maxwell distribution under different loss functions. *Journal of Statistical Theory and Practice*, 4(2), 279-287.
- Dey, S., Dey, T., and Maiti, S. S. (2013). Bayesian inference for Maxwell distribution under conjugate prior. *Model Assisted Statistics and Applications*, 8(3), 193-203.
- Eraikhuemen, I. B., Bamigbala, O. A., Magaji, U. A., Yakura, B. S. and Manju, K. A. (2020). Bayesian analysis of Weibull-Lindley distribution using different loss functions, *Asian Journal of Advanced Research and Reports*, 28–41.
- Eraikhuemen, I. B., Mohammed, F. B. and Sule, A. A. (2020). Bayesian and Maximum Likelihood Estimation of the Shape Parameter of Exponential Inverse Exponential Distribution: A Comparative Approach, *ØAsian Journal of Probability and Statistics*, 28–43.
- Feroze, N. (2012). Estimation of scale parameter of inverse gaussian distribution under a Bayesian framework using different loss functions. *ØScientific Journal of Review*, 1(3); 39-52.
- Feroze, N. and Aslam, M. (2012). A note on Bayesian analysis of error function distribution under different loss functions. *ØInternational Journal of Probability and Statistics*, 1(5); 153-159.
- Gupta, I. (2017). Estimation of Parameter and Reliability Function of Exponentiated Inverted Weibull Distribution using Classical and Bayesian Approach. *ØInternational Journal of Recent Scientific Research*, 8(7): 18117-18819.

- Gupta, P.K. and Singh, A. K. (2017). Classical and Bayesian estimation of Weibull distribution in presence of outliers. ØCogent Mathematics, 4: 1300975.
- Himanshu, P. and Arun, K. R. (2009). Bayesian Estimation of the Shape Parameter of Generalized Pareto Distribution Under Asymmetric Loss Functions. ØHacettepe Journal of Mathematics and Statistics, 38 (1), 69-83.
- Ieren, T. G. and Chukwu, A. U. (2018), Bayesian analysis of a shape parameter of the Weibull-Frechet distribution, ØAsian Journal of Probability and Statistics, 1–19,
- Ieren, T. G. and Oguntunde, P. E. (2018). A Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution. ØAsian Journal of Probability and Statistics, 1(1): 1-12.
- Mabur, T. M., Omale, A., Lawal, A., Dewu, M. M. and others (2018). Bayesian estimation of the scale parameter of the Weibull distribution, ØAsian Journal of Probability and Statistics, 1–9.
- Martz, H. F., & Waller, R. A. (1982). Bayesian reliability analysis. ØNew York, NY: John Wiley.
- Nasir, W., Maria, and Aslam, M. (2015). Bayes approach to study scale parameter of log logistic distribution. ØInternational Journal of Advanced Statistics and Probability, 3(2); 185-190
- Navid, F. and Muhammad, A. (2012). A Note on Bayesian Analysis of Error Function Distribution under Different Loss Functions. ØInternational Journal of Probability and Statistics, 1 (5): 153-159.
- Norstrom, J. G. (1996). The use of Precautionary Loss Functions in Risk Analysis, ØIEEE Transactions on reliability, 3: 400-403.
- Preda, V., Eugenia, P. and Alina, C. (2010). Bayes Estimators of Modified-Weibull Distribution parameters using Lindley's approximation. ØWSEAS TRANSACTIONS on MATHEMATICS, 9 (7), 539-549.
- Sajid, A., Muhammad, A., Nasir, A. and Syed, M. A. (2012). Scale Parameter Estimation of the Laplace Model Using Different Asymmetric Loss Functions. ØInternational Journal of Statistics and Probability, 1 (1), 105-127.
- Sanku, D. (2010). Bayesian Estimation of the Shape Parameter of the Generalised Exponential Distribution under Different Loss Functions. ØPakistan Journal of statistics and Operations Research, VI (2), 163-174.
- Sanku, D. and Tanujit, D. (2011). Rayleigh Distribution; Revisited via extension of Jeffrey's prior information and a new loss function. ØStatistical Journal, IX (3), 213-226.
- Singpurwalla, N. D. (2006). Reliability and risk: A Bayesian perspective. ØChichester: John Wiley.
- Tahir, M. H., Cordeiro, G.M., Mansoor M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications. ØHacettepe Journal of Mathematics and Statistics. Volume 44 (2) (2015), 461 – 480
- Wajiha, N. and Muhammed, A. (2015). Bayes approach to study shape parameter of Frechet distribution. ØInternational Journal of Basic and Applied Sciences, 4 (3), 246 – 254
- Wasif, Y. and Navid, F. (2015). Posterior Analysis of Nakagami Distribution Under Different Loss Functions. ØScience international (Lahore), 27 (3), 1819-1824.