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Alpha Level and Inverse Alpha Level Of Multi-Fuzzy Set

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Abstract

In this paper, the concepts of inverse alpha level and weak inverse alpha level of multi-fuzzy set together with their properties are introduced. It is shown that the inverse alpha level is always contained in the weak inverse alpha level. It is further demonstrated that as the union of the alpha levels of two multi-fuzzy sets is contained in the alpha level of their union, it is only the converse that is obtainable for the case of inverse alpha level as well as weak inverse alpha level. Moreover, related results are also provided.

Key words: Fuzzy Set, Multiset, Multi-fuzzy Set, Alpha level mset

1. Introduction

The theories of multisets and fuzzy sets were introduced as a result of the discomfort expressed by various scholars as presented in (Singh et al., 2007) on the too restrictive demand of distinctness and definiteness on the definition of classical set. Relaxing the restriction of distinctness on the nature of the objects forming a set gave rise to multiset theory. The reader may refer to (Blizard, 1989; Singh et al., 2007; Yager, 1986) for more details on the concept of multisets. Moreover, by way of relaxing the restriction of definiteness imposed on objects to form an ordinary set, fuzzy set theory was formulated. It is a mathematical theory introduced to model vagueness and other loose concepts. In order to comprehend the notion of loose concepts like classes with vague boundaries, etc., Zadeh (1965) suggests noting that the notion of belonging in such cases is not the same as it has in ordinary sets. A fuzzy set is a generalized set of objects occurring with a continuum of degrees (grades) of membership. Intuitively, fuzzy multisets model the case where otherwise indistinguishable objects possess a particular property to a certain degree. If we go the other way and fuzzify the number of occurrences of each object, then we get a new structure, which we call a multi-fuzzy set. The motivating factor for introducing this was to fuzzify the basic property of multisets and, thus, to define multisets where the number of occurrences forms a fuzzy set (Blizard, 1989). Many researchers such as (Blizard, 1991; Singh et al., 2008; Kosko, 1992) have contributed to the developments of both the theories.
The notion of α-level set or α-cut of a fuzzy set was first introduced in (Zadeh, 1965) while Sun and Han (2006) introduced the notion of an inverse α-cut of a fuzzy set and described its properties. Moreover, Singh et al., (2014) studied analogues of properties of α-cuts and Decomposition theorems of fuzzy sets in fuzzy multisets. The concept of alpha cut and n-level

sets were studied in (Isah, 2019; 2019a), (Singh et al., 2015), (Isah et al. 2019) and (Alkali and Isah, 2019).

2. Preliminaries

2.1 Fuzzy Sets

Definition 2.1.1 [Zadeh, 1965; Klir and Yuan, 1995]
A fuzzy set (class) $\tilde{A}$ over the set X is characterized by a membership function $\mu_{\check{A}}(x)$ which associates with each point x in X, a real number $\mu_{\check{A}}(x)$ in the interval $[0, 1]$. The value of $\mu_{\check{A}}(x)$ represents the grade of membership of x in $\check{A}$.

Let $\check{A}$ and $\check{B}$ be two fuzzy sets, then

(a) $\check{A} \subseteq \check{B}$ if and only if
$\mu_{\check{A}}(x) \leq \mu_{\check{B}}(x), \quad \forall x \in X$.

(b) $\check{A} \cup \check{B} = \check{C}$ such that
$$\mu_{\check{C}}(x) = \max\{\mu_{\check{A}}(x), \mu_{\check{B}}(x)\}, \forall x \in X.$$

(c) $\check{A} \cap \check{B} = \check{C}$ such that
$$\mu_{\check{C}}(x) = \min\{\mu_{\check{A}}(x), \mu_{\check{B}}(x)\}, \forall x \in X.$$

Definition 2.1.2 [Zadeh, 1965]
Let X be a non-empty set and $F(X)$ the set of all fuzzy sets over the set X. Let $A \in F(X)$ and $\alpha \in [0,1]$. Then the non-fuzzy set (or crisp set)
$${}^{\alpha}A = \{x \in X | \mu_A(x) \geq \alpha\}$$
is called the *α-cut* or *α-level* set of A.
The *strong α-cut*, denoted by ${}^{\alpha+}A$ is defined as
$${}^{\alpha+}A = \{x \in X | \mu_A(x) > \alpha\}.$$

Definition 2.1.3 [Sun and Han, 2006]
Let $A \in F(X)$ and $\alpha \in [0,1]$. Then the non-fuzzy set ${}^{\alpha-}A^{-1} = \{x \in X | \mu_A(x) < \alpha\}$ is called a strong *inverse α -cut* or a strong *inverse α -level set* of A.
The *weak inverse α-cut* or alpha level of A, denoted by ${}^{\alpha}A^{-1}$ is defined as
$$({}^{\alpha}A^{-1}) = \{x \in X | \mu_A(x) \leq \alpha\}.$$

2.2 Multisets

Definition 2.2.1 [Blizard, 1991; Singh et al., 2008]
Let $X = \{x_1, x_2, x_3, \cdots, x_j, \cdots\}$ be set. A multiset (or mset for short) A over the set X is a *cardinal-valued* function i.e., $A: X \to N = \{0,1,2, \cdots\}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of times an object x occurs in A.
Let A be a mset containing one occurrence of a, two occurrences of b, and three occurrences of c, then A can be represented as $A = [[a, b, b, c, c, c]] = [a, b, b, c, c, c] = [a, b, c]_{1,2,3} = [a, 2b, 3c] = [a.1, b.2, c.3] = [1/a, 2/b, 3/c] = [a^1, b^2, c^3] = [a^1 b^2 c^3]$. For convenience, the curly brackets are also used in place of the square brackets.

Definition 2.2.2 Let A and B be two msets over a given domain set X. Then

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(a) $A \subseteq B$ if $m_A(x) \leq m_B(x), \forall x \in X$.
(b) A is said to be a whole submset of B if $A \subseteq B \wedge m_A(x) = m_B(x), \forall x \in X$.
(c) $A = B$ if $m_A(x) = m_B(x), \forall x \in X$.
(d) $A \cup B = C$ such that

$$m_C(x) = \max\{m_A(x), m_B(x)\}, \forall x \in X.$$

(e) $A \cap B = C$ such that

$$m_C(x) = \min\{m_A(x), m_B(x)\}, \forall x \in X.$$

(f) $A + B = A \uplus B = C$ such that

$$m_C(x) = m_A(x) + m_B(x), \forall x \in X.$$

(g) $A - B = C$ such that

$$m_C(x) = \max\{m_A(x) - m_B(x), 0\}, \forall x \in X.$$

2.3 Multi-fuzzy sets [Syropoulos, 2006]

Definition 2.3.1 Let X be auniversal set, then a multi-fuzzy set (m-fuzzy set, for short) A is a function $A: X \to N_0 \times I$, defined by $A(x) = (n, i)$ where N_0 is the set of all positive integers including zero, $I = [0, 1]$, $n \in N_0, i \in I$and (n, i) denotes that the degree to which x occurs n times in the m-fuzzy set A is equal to i.

Let A be a m-fuzzy set, then

(a) the multiplicity function for A denoted A_m is a function$A_m: X \to N_0$ and
(b) the degree of membership function is defined as $\mu_A: X \to I$.

Thus$A(x) = (n, i)$, implies$A_m(x) = n$ and $\mu_A(x) = i$ i.e., $A(x) = \left(A_m(x), \mu_A(x)\right)$.

Definition 2.3.2 Let A be a m-fuzzy set over a set X, then we say $x \in A \Leftrightarrow A_m(x), \mu_A(x) > 0$ and $x \notin A \Leftrightarrow A_m(x), \mu_A(x) = 0$. Moreover, A is called an empty m-fuzzy set if $A_m(x), \mu_A(x) = 0, \ \forall x \in X$.

Definition 2.3.3 Let A and B be two m-fuzzy sets over a set X, then we have

Equality:

$$A = B \Leftrightarrow A_m(x) = B_m(x) \wedge \mu_A(x) = \mu_B(x), \forall x \in X.$$

Subm-fuzzy set:

$$A \subseteq B \Leftrightarrow A_m(x) \leq B_m(x) \wedge \mu_A(x) \leq \mu_B(x), \forall x \in X.$$

Thus, $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

Union: $A \cup B$ is such that

$$(A \cup B)(x) = (\max\{A_m(x), B_m(x)\}, \max\{\mu_A(x), \mu_B(x)\}), \forall x \in X.$$

Intersection: $A \cap B$ is such that
$(A \cap B)(x) = (\min\{A_m(x), B_m(x)\}, \min\{\mu_A(x), \mu_B(x)\}), \forall x \in X.$

Sum: $A \uplus B$ is such that

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$$(A \uplus B)(x) = (A_m(x) + B_m(x) , (\mu_A + \mu_B)(x)), \forall x \in X,$$
where $(\mu_A + \mu_B)(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$.

Difference:
$$A - B = (\max\{A_m(x) - B_m(x), 0\}, \max\{\mu_A(x) - \mu_B(x), 0\}), \forall x \in X.$$

Example 2.3.4
Let $A = \{(3,0.5)/x, (5,0.7)/y, (2,0.4)/z\}$ and $B = \{(2,0.2)/x, (4,0.8)/y, (5,0.3)/z\}$ be two m-fuzzy sets over the set $X = \{x, y, z\}$. Then

$$A \cup B = \{(3,0.5)/x, (5,0.8)/y, (5,0.4)/z\}$$

$$A \cap B = \{(2,0.2)/x, (4,0.7)/y, (2,0.3)/z\}$$
$$A \uplus B = \{(5,0.6)/x, (9,0.94)/y, (7,0.58)/z\}.$$

3. Alpha Level of Multi-Fuzzy Set

Definition 3.1 Let A be a m-fuzzy set over X and $\alpha \in [0,1]$, then the alpha level or α −cut of A is the mset $[A]_\alpha$ defined by $[A]_\alpha = \{A_m(x)/x : \mu_A(x) \geq \alpha, \forall x \in X\}$.

The strong α −level of A is the mset $[A]_{\alpha+}$ defined by $[A]_{\alpha+} = \{A_m(x)/x : \mu_A(x) > \alpha, \forall x \in X\}$.

Example 3.2 Let $X = \{a, b, c, d\}$ and $A = \{(3,0.5)/a, (5,0.6)/b, (4,0.3)/c, (6,0.7)/d\}$, then

$$[A]_{0.3} = [3/a, 5/b, 4/c, 6/d], [A]_{0.5} = [3/a, 5/b, 6/d], [A]_{0.6} = [5/b, 6/d], [A]_{07} = [6/d].$$

$$[A]_{0.3+} = [3/a, 5/b, 6/d], \quad [A]_{0.5+} = [5/b, 6/d], \quad [A]_{0.6+} = [\ 6/d], \quad [A]_{07+} = \emptyset.$$

Proposition 3.3 Let A and B be m-fuzzy sets over the set X and $\alpha \in [0,1]$, then we have

i. $[A]_{\alpha+} \subseteq [A]_\alpha$and $[A]_{\alpha+}$ is whole in $[A]_\alpha$
ii. $A \subseteq B \Longrightarrow [A]_\alpha \subseteq [B]_\alpha$
iii. $A = B \Longrightarrow [A]_\alpha = [B]_\alpha$
iv. If $\alpha_1 \leq \alpha_2 \Longrightarrow [A]_{\alpha_2} \subseteq [A]_{\alpha_1}$
v. $[A]_\alpha \cup [B]_\alpha \subseteq [A \cup B]_\alpha$
vi. $[A \cap B]_\alpha = [A]_\alpha \cap [B]_\alpha$
vii. $[A \cap B]_{\alpha+} = [A]_{\alpha+} \cap [B]_{\alpha+}$

Proof

i. Let $A_m(x)/x \in [A]_{\alpha+}$ for any $x \in X$. Then $\mu_A(x) > \alpha$ by definition.

$\Longrightarrow \mu_A(x) \geq \alpha$

$\Longrightarrow A_m(x)/x \in [A]_\alpha$

i.e., $[A]_{\alpha+} \subseteq [A]_\alpha$

Moreover, $\forall x \in X,\ A_m(x)/x \in [A]_{\alpha+} \Longrightarrow A_m(x)/x \in [A]_\alpha$.

This implies that $A_m(x)/x$ is the same in both $[A]_{\alpha+}$ and $[A]_\alpha$, and as $[A]_{\alpha+} \subseteq [A]_\alpha$ implies $[A]_{\alpha+}$ is whole in $[A]_\alpha$.

Proofs of (ii) and (iii) follows from definitions 3.1 and 2.3.3.

iv. Let $\alpha_1 \leq \alpha_2$ and let $A_m(x)/x \in [A]_{\alpha_2}$ for any $x \in X$. Then $\mu_A(x) \geq \alpha_2$ by definition.

Since $\alpha_1 \leq \alpha_2$, then$\mu_A(x) \geq \alpha_2 \geq \alpha_1$. In particular, $\mu_A(x) \geq \alpha_1$ and $A_m(x)/x \in [A]_{\alpha_1}$.

Thus, $[A]_{\alpha_2} \subseteq [A]_{\alpha_1}$

v. Let $C_m(x)/x \in [A]_\alpha \cup [B]_\alpha$. Then by definition

$$C_m(x)/x = (max\{A_m(x), B_m(x)\}/x\,, max\{\mu_A(x), \mu_B(x)\} | \mu_A(x), \mu_B(x) \geq \alpha)$$

In particular, $max\{\mu_A(x), \mu_B(x)\} \geq \alpha$ i.e.,$\mu_{A\cup B}(x) \geq \alpha$

and $max\{A_m(x), B_m(x)\}/x = (A_m \cup B_m)(x)/x$.

Thus, $C_m(x)/x = ((A_m \cup B_m)(x)/x\,, \mu_{A\cup B}(x) \wedge \mu_{A\cup B}(x) \geq \alpha)$

i.e., $C_m(x)/x = (A \cup B)(x)/x \wedge \mu_{A\cup B}(x) \geq \alpha$.

In particular, $C_m(x)/x \in [A \cup B]_\alpha$ and $[A]_\alpha \cup [B]_\alpha \subseteq [A \cup B]_\alpha$.

vi. Let $C_m(x)/x \in [A]_\alpha \cap [B]_\alpha$ i.e., $C_m(x)/x \in [A]_\alpha$ and $C_m(x)/x \in [B]_\alpha$. Then by definition

$$C_m(x)/x = (min\{A_m(x), B_m(x)\}/x\,, min\{\mu_A(x), \mu_B(x)\} | \mu_A(x), \mu_B(x) \geq \alpha)$$

In particular, $min\{\mu_A(x), \mu_B(x)\} \geq \alpha$ i.e., $\mu_{A\cap B}(x) \geq \alpha$

and $min\{A_m(x), B_m(x)\}/x = (A_m \cap B_m)(x)/x$.

Thus, $C_m(x)/x = ((A_m \cap B_m)(x)/x\,, \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(x) \geq \alpha)$

i.e., $C_m(x)/x = (A \cap B)(x)/x \wedge \mu_{A\cap B}(x) \geq \alpha$.

In particular, $C_m(x)/x \in [A \cap B]_\alpha$ and $[A]_\alpha \cap [B]_\alpha \subseteq [A \cap B]_\alpha$. …….. (i)

Conversely, let$C_m(x)/x \in [A \cap B]_\alpha$. Then by definition

$$C_m(x)/x \in (min\{A_m(x), B_m(x)\}/x\,, min\{\mu_A(x), \mu_B(x)\} | \mu_A(x), \mu_B(x) \geq \alpha)$$

$C_m(x)/x \in \{A_m(x)/x : \mu_A(x) \geq \alpha\}$ and $C_m(x)/x \in \{B_m(x)/x : \mu_B(x) \geq \alpha\}$

$C_m(x)/x \in [A]_\alpha$ and $C_m(x)/x \in [B]_\alpha$ i.e., $C_m(x)/x \in [A]_\alpha \cap [B]_\alpha$

i.e.,$[A \cap B]_\alpha \subseteq [A]_\alpha \cap [B]_\alpha$………………….... (ii)

From (i) and (ii) we have $[A \cap B]_\alpha = [A]_\alpha \cap [B]_\alpha$.

4. Inverse Alpha Level of Multi-Fuzzy Set

Definition 4.1 Let A be a multi-fuzzy (m-fuzzy) set over the set X and $\alpha \in [0,1]$, then the multiset $[A]_{\alpha-}^{-1} = \{A_m(x)/x : \mu_A(x) < \alpha, \forall x \in X\}$ is called the inverse $\alpha -$ level of A.

The multiset (mset) $[A]_\alpha^{-1} = \{A_m(x)/x : \mu_A(x) \leq \alpha, \forall x \in X\}$ is called the weak inverse $\alpha -$ level of A.

Example 4.2 Let $X = \{x_1, x_2, x_3, x_4\}$ and $A = \{(3,0.2)/x_1, (6,0.5)/x_2, (4,0.3)/x_3, (5,1)/x_4\}$, then

$$[A]_{0.1-}^{-1} = \emptyset, \quad [A]_{0.2-}^{-1} = \emptyset, \quad [A]_{0.3-}^{-1} = [3/x_1], \quad [A]_{0.4-}^{-1} = [A]_{0.5-}^{-1} = [3/x_1, 4/x_3],$$

$$[A]_{0.6-}^{-1} = [A]_{0.7-}^{-1} = [A]_{0.8-}^{-1} = [A]_{0.9-}^{-1} = [A]_{1-}^{-1} = [3/x_1, 6/x_2, 4/x_3].$$

$$[A]_{0.1}^{-1} = \emptyset, \quad [A]_{0.2}^{-1} = [3/x_1], \quad [A]_{0.3}^{-1} = [A]_{0.4}^{-1} = [3/x_1, 4/x_3],$$
$$[A]_{0.5}^{-1} = [A]_{0.6}^{-1} = [A]_{0.7}^{-1} = [A]_{0.8}^{-1} = [A]_{0.9}^{-1} = [3/x_1, 6/x_2, 4/x_3],$$
$$[A]_1^{-1} = [3/x_1, 6/x_2, 4/x_3, 5/x_4]$$

Remark 4.3

Contrary to proposition 3.3 (vi), $[A]_{\alpha-}^{-1} \cap [B]_{\alpha-}^{-1} = [A \cap B]_{\alpha-}^{-1}$ fails.

Example

Consider a m-fuzzy set A above, and let

$B = \{(3,0.4)/x_1, (4,,0.1)/x_2, (1,0.7)/x_3, (2,0.3)/x_4\}$, then

$[A]_{0.3-}^{-1} \cap [B]_{0.3-}^{-1} = \emptyset$ while $[A \cap B]_{0.3-}^{-1} = [3/x_1, 4/x_2]$.

Remark 4.4

$[A]_{\alpha-}^{-1} \cup [B]_{\alpha-}^{-1} = [A \cup B]_{\alpha-}^{-1}$ do not hold.

Example

$[A]_{0.7-}^{-1} \cup [B]_{0.7-}^{-1} = [3/x_1, 6/x_2, 4/x_3, 2/x_4]$ while $[A \cup B]_{0.7-}^{-1} = [3/x_1, 6/x_2]$.

Proposition 4.5 Let A and B be m-fuzzy sets over the set X and $\alpha \in [0,1]$. Then

i. $[A]_{\alpha-}^{-1} \subseteq [A]_{\alpha}^{-1}$
ii. If$\alpha_1 \leq \alpha_2 \Longrightarrow [A]_{\alpha_1-}^{-1} \subseteq [A]_{\alpha_2-}^{-1}$ and $[A]_{\alpha_1}^{-1} \subseteq [A]_{\alpha_2}^{-1}$
iii. $[A \cup B]_{\alpha-}^{-1} \subseteq [A]_{\alpha-}^{-1} \cup [B]_{\alpha-}^{-1}$
iv. $[A \cup B]_{\alpha}^{-1} \subseteq [A]_{\alpha}^{-1} \cup [B]_{\alpha}^{-1}$
v. $[A]_{\alpha}^{-1} \cap [B]_{\alpha}^{-1} \subseteq [A \cap B]_{\alpha}^{-1}$
vi. $[A]_{\alpha-}^{-1} \cap [B]_{\alpha-}^{-1} \subseteq [A \cap B]_{\alpha-}^{-1}$

Proof

Proofs of (i) and (ii) follows from definition 4.1.

iii. Let$C_m(x)/x \in [A \cup B]_{\alpha-}^{-1}$. Then by definition

$\Longrightarrow C_m(x)/x \in (max\{A_m(x), B_m(x)\}/x\,, max\{\mu_A(x), \mu_B(x)\}|\mu_A(x), \mu_B(x) < \alpha)$

i.e., $C_m(x)/x \in \{A_m(x)/x : \mu_A(x) < \alpha\}$ or $C_m(x)/x \in \{B_m(x)/x : \mu_B(x) < \alpha\}$

$\Longrightarrow C_m(x)/x \in [A]_{\alpha-}^{-1}$ or $C_m(x)/x \in [B]_{\alpha-}^{-1}$ i.e., $C_m(x)/x \in [A]_{\alpha-}^{-1} \cup [B]_{\alpha-}^{-1}$

i.e., $[A \cup B]_{\alpha-}^{-1} \subseteq [A]_{\alpha-}^{-1} \cup [B]_{\alpha-}^{-1}$.

The proof of (iv) is analogous to that of (iii)'

v. Let $C_m(x)/x \in [A]_{\alpha}^{-1} \cap [B]_{\alpha}^{-1}$

i.e., $C_m(x)/x \in [A]_{\alpha}^{-1}$ and $C_m(x)/x \in [B]_{\alpha}^{-1}$. Thus by definition

$$C_m(x)/x = (min\{A_m(x), B_m(x)\}/x\,, min\{\mu_A(x), \mu_B(x)\}|\mu_A(x), \mu_B(x) \leq \alpha)$$

In particular, $min\{\mu_A(x), \mu_B(x)\} \leq \alpha$i.e., $\mu_{A\cap B}(x) \leq \alpha$ and

$min\{A_m(x), B_m(x)\}/x = (A_m \cap B_m)(x)/x$.

Thus, $C_m(x)/x = ((A_m \cap B_m)(x)/x\,, \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(x) \leq \alpha)$

i.e., $C_m(x)/x = (A \cap B)(x)/x \wedge \mu_{A\cap B}(x) \leq \alpha$.

i.e., $C_m(x)/x \in [A \cap B]_{\alpha}^{-1}$ and $[A]_{\alpha}^{-1} \cap [B]_{\alpha}^{-1} \subseteq [A \cap B]_{\alpha}^{-1}$.

The proof of (vi) is analogous to that of (v).

5. Conclusion

The paper presents the concepts of inverse α-level and weak inverse α-level of m-fuzzy sets together with some of their properties. It is also established that as some properties are satisfied in inverse α-level, they actually fails in the weak inverse α-level.

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