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A Dissection of Rhotrix Ring Into Their Subring And Properties

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Abstract
This paper presents a study of algebraic structure of certain ring, whose underlying set is the rhotrix set; consisting of all real rhotrices of the same size over real numbers in order to uncover some properties of ring of real rhotrices and its subrings. Furthermore, we dissect some of these properties in order to establish isomorphic relationship between some of them, in form of lemmas and theorems while adopting heart to heart rhotrix multiplication.

Keyword: Rhotrix, Rhotrix Set, Ring of Rhotrices, Isomorphic

1. Introduction

New development in algebra has not only given way to new concept but an entire new looks to Mathematics along with new proofs of difficult classical results. The analysis of a problem into its basic concept has often uncovered the proper settings of the problem in order to explore the system effectively by relating it with an existing problem. In this study we present algebraic structure of certain ring, whose underlying set is the rhotrix set; consisting of all real rhotrices of the same size over real numbers in order to uncover some properties of ring of real rhotrices and its subrings. The ring of rhotrix under study is a combination of three items $(R^*,+,\bullet)$, i.e. a rhotrix set having rhotrices of the same size as entries on which two operations, of addition and multiplication operates with a certain rule of combination, provided these operations satisfy ring axioms or postulates. Thus, $(R^*,+)$ satisfies the axioms for group while $(R^*,+,\bullet)$ satisfies the axioms of rings (Sani, 2004). Rhotrix is a mathematical object that is interwoven between $(2\times 2)-$ dimensional matrices and $(3\times 3)-$ dimensional matrices, presented as an extension of ideas on matrix-tertion and matrix-noitret, (Ajibade, 2003) and (Atanassov, and Shannon, 1998). An alternative method for multiplication of rhotrix and some relationships between rhotrix and matrices were established (Sani, 2004). The row-column multiplications of higher dimensional rhotrices were also presented in (Sani, 2007), as an extension of the same multiplication carried out on rhotrix of dimensions three which is considered to be the base rhotrix. An outline of enrichment exercise of rhotrices was further presented to stimulate ideas on their mathematical arrays and their classification as an abstract structure (Muhammed, 2007a). Additional classifications of rhotrices as abstract structures of ring, field, integral domain, principal ideal domain and unique factorization domain were presented as a sort of additional work to an earlier classification of rhotrix as Algebraic structures of groups, semi groups, monoids and Boolean algebra, (Mohammed, 2009). A study on rhotrix sets and rhotrix spaces over the field of numbers were carried-out and further stimulate the idea of systematization of characterizing rhotrix spaces over real fields. (Muhammed and Tella, 2012), Following the adoption of column multiplication in rhotrices, a non-commutative rhotrix ring and its subring were presented by (Mohammed, 2018).

2. Materials and Method

Initial Definition of a Rhotrix Set

A set of all rhotrices of the size as three was defined in (Ajibade, 2003) as

$$R^{\Diamond}(3,\Re)=\left\{\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array}\right\rangle : a,b,c,d,e\in\Re\right\} \qquad (1)$$

where $\mathrm{h}(R) = c$ is called the heart of any rhotrix $R\in R^{\Diamond}(3,\Re)$. The rhotrices in $R^{\Diamond}(3,\Re)$ are all of the same size *n=3*, with five element as entries each. So, for any rhotrix, $R\in R^{\Diamond}(3,\Re),$ the weight of *R* is $|R|=5.$

Generalized Definition of a Rhotrix Set

A generalized definition of a rhotrix set $R^{\Diamond}(n,\Re)$, as a mathematical rhomboid array of the form was presented by (Mohammed, and Tijjani, 2011).as:

$$R^{*}(n,\Re)=\left\{\left\langle \begin{array}{ccccccccc} & & & & r_1 & & & & \\ & & r_2 & & r_3 & & r_5 & & \\ & - & - & & - & & - & - & \\ - & - & - & & r_{\frac{1}{4}(n^2+3)} & & - & - & - \\ & - & - & & - & & - & - & \\ & & r_{\frac{1}{2}(n^2+1)-3} & & r_{\frac{1}{2}(n^2+1)-2} & & r_{\frac{1}{2}(n^2+1)-1} & & \\ & & & & r_{\frac{1}{2}(n^2+1)} & & & & \end{array}\right\rangle : r_1, r_2, \ldots, r_{r_{\frac{1}{2}(n^2+1)}} \in\Re\right\}, \qquad (2)$$

where $n \in 2Z^{+} + 1$ and $h(R) = \frac{1}{4}(n^2 + 3)$ is called the heart of any rhotrix $R(n) \in R^{\Diamond}(n, \Re)$. Thus, the weight of any rhotrix $R(n) \in R^{\Diamond}(n, \Re)$ is given by $|R| = \frac{1}{2}\left(n^2 + 1\right)$ and $n \in (2Z^{+} + 1)$.

Identity Rhotrix

If for any two rhotrices R and I, we have $R \, o \, I = I \, o \, R = R$ then I is called the identity element in the rhotrix set. i.e,

$$\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \bullet \left\langle \begin{array}{ccc} & f & \\ g & h(I) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \qquad (3)$$

Equation (3) yields

$$\left.\begin{array}{c} ah(I) + fh(R) = a \\ bh(I) + gh(R) = b \\ h(R)h(I) = h(R) \\ dh(I) + jh(R) = d \\ eh(I) + kh(R) = e \end{array}\right\} \qquad (4)$$

Solving equation (4), where $f = g = j = k = 0$, $h(I) = 1$, so that

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

is the identity rhotrix.

Inverse of a Rhotrix

If for any two rhotrices R and P, we have R o P = P o R = I, then P is the inverse of R. that is,

$$\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \bullet \left\langle \begin{array}{ccc} & f & \\ g & h(P) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \qquad (5)$$

Equation (5) Yields

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$$\left.\begin{aligned} ah(P) + fh(R) &= 0 \\ bh(P) + gh(R) &= 0 \\ h(R)h(P) &= 1 \\ dh(P) + jh(R) &= 0 \\ eh(P) + kh(R) &= 0 \end{aligned}\right\} \quad (6)$$

Solving equation (6), we have
h(P) = 1/h(R), provided h(R) ≠ 0 and f = -a/(h(R))^2, g = -b/(h(R))^2,
j = -d/(h))^2, k = -e/(h(R))^2 , so that

$$P = R^{-1} = -\frac{1}{(h(R)^2}\left\langle \begin{matrix} & a & \\ b & -h(R) & d \\ & d & \end{matrix} \right\rangle$$ is the inverse rhotrix.

Real Rhotrix and Real Rhotrix Set

The definition of real rhotrix and real rhotrix set was presented by (Mohammed, 2009) and (Muhammed and Tella, 2012).

A rhotrix *R*, is called a real rhotrix if all its entries belong to set of real numbers.

A rhotrix set $\boldsymbol{R}^{\Diamond}$, is called a real rhotrix set if all of its elements are real rhotrices.

Following (Sani, 2004), a constructed ring of all real rhotrices of the same size was present as follows: -

Let $\boldsymbol{R}^{*} = \langle \boldsymbol{R}^{\Diamond}, +, \bullet \rangle$ be an abstract structure consisting of the set of real rhotrices of the same size, where the operations of addition and multiplication are also defined.

Thus, it is noteworthy to mention that the algebraic system $\boldsymbol{R}^{*} = \langle \boldsymbol{R}^{\Diamond}, +, \bullet \rangle$ satisfies all the six axioms of a ring.

$$R^{\Diamond}(3, \Re) = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \Re \right\}$$

3. Results and Discussion

We now dissect the rhotrix ring initiated in (Mohammed, 2009) and (Muhammed,2008) in order to identify its subrings. Furthermore, a number of useful results are drawn.

Ring of all zero heart rhotrices of the same size

A rhotrix *R*, whose heart *h(R)*, is equal to zero is called a zero heart rhotrix. The set of all zero heart rhotrices of the same size, together with the rhotrix operations of addition and multiplication is a ring. This ring may be called ring of all zero heart rhotrices.

Ring of all integer rhotrices of the same size

A rhotrix *R*, whose entries belong to the set of integer numbers, is called integer rhotrix. The set of all integer rhotrices of the same size, together with the rhotrix operations of addition and multiplication is a ring. This ring may be called ring of all integer rhotrices of the same size denoted by $Z^* = \langle Z^\diamond, +, \bullet \rangle$.

Ring of all rational rhotrices of the same size

A rhotrix *R*, whose entries belong to the set of rational numbers, is called rational rhotrix. The set of all rational rhotrices of the same size, together with the rhotrix operations of addition and multiplication is a ring. This ring may be called ring of all rational rhotrices of the same size denoted by $R^* = \langle R^\diamond, +, \bullet \rangle$.

Proposition

If $Z^* = \langle Z^\diamond, +, \bullet \rangle$ is a ring of all integer rhotrices of the same size *3* then Z^* is a subring of R^*, the ring of all real rhotrices of the same size *3*.

Proof

Let the set of all integer rhotrices of size three be define as

$$Z^\diamond = \left(\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in Z \right).$$

Since, $Z^* = \langle Z^\diamond, +, \bullet \rangle$ clearly, Z^* is non-empty. Also, it is simple to show that Z^* is closed under the operations of addition '+'and multiplication $'\bullet'$ from R^*. Furthermore, Z^* contains the identity element I_{R^*} in R^*. Hence, Z^* is a subring of R^*.

Proposition

If $Q^* = \langle Q^\diamond, +, \bullet \rangle$ is a ring of all rational rhotrices of the same size *3* then Q^* is a subring of R^*, the ring of all real rhotrices of the same size *3*.

Proof

Let the set of all rational rhotrices of size three be define as

$$Q^\diamond = \left(\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in Q \right).$$

Since, $Q^* = \langle Q^\diamond, +, \bullet \rangle$ clearly, Q^* is non-empty. Also, it is simple to show that Q^* is closed under the operations of addition '+' and multiplication $'\bullet'$ from R^*. Furthermore, Q^* contains the identity element I_{R^*} in R^*. Hence, Q^* is a subring of R^*.

Theorem

In a ring of all real rhotrices of the same size, there exists a chain of subrings

$\langle Z^\diamond, +, \bullet \rangle \subset \langle Q^\diamond, +, \bullet \rangle \subset \langle R^\diamond + , \bullet \rangle$, i.e. $Z^* \subset Q^* \subset R^*$

Proof

It well known that, the set of all integer numbers is a subset of the set of all rational numbers, and the set of all rational numbers is contained in the set of all real numbers. Also, it follows that, $Z^\diamond \subset Q^\diamond \subset R^\diamond$. Since, $Z^* = \langle Z^\diamond, +, \bullet \rangle$ and $Z^* = \langle Z^\diamond, +, \bullet \rangle$ are subrings of $R^* = \langle R^\diamond, +, \bullet \rangle$ as shown in above propositions.

Hence, $Z^* \subset Q^* \subset R^*$.

Lemma

If $O^{*} = \langle O^{\Diamond}, +, \bullet \rangle$ is a ring of all zero-heart rhotrices of the same size three then O^{*} is a subring of R^{*}, the ring of all real rhotrices of the same size three.

Proof

Let the set of all zero heart rhotrices of size three be define as

$$O^{\Diamond} = \left(\left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle : a, b, 0, d, e \in \Re \right).$$

Since, $O^{*} = \langle O^{\Diamond}, +, \bullet \rangle$ clearly, O^{*} is non-empty. Also, it is simple to show that O^{*} is closed under the operations of addition '+' and multiplication '$\bullet$' from R^{*}. Furthermore, O^{*} contains the identity element $I_{R^{*}}$ in R^{*}. Hence, O^{*} is a subring of R^{*}.

Theorem

The ring $Z^{*} = \langle Z^{\Diamond}, +, \bullet \rangle$, of all integer rhotrices of size three is isomorphic to the ring of all twice $2Z^{*} = \langle 2Z^{\Diamond}, +, \bullet \rangle$ integer rhotrices of size three.

Proof

Let the set of all integer rhotrices of size three be define as

$$Z^{\Diamond} = \left(\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in Z \right).$$

Then it follows that

$$2Z^{\Diamond} = \left(\left\langle \begin{matrix} & 2a & \\ 2b & 2c & 2d \\ & 2e & \end{matrix} \right\rangle : a, b, c, d, e \in Z \right).$$

Now, let $Z^{*} = \langle Z^{\Diamond}, +, \bullet \rangle$ and $2Z^{*} = \langle 2Z^{\Diamond}, +, \bullet \rangle$. It is easy to show that, Z^{*} and $2Z^{*}$ are rings of rhotrices having the same order. We define the mapping

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$\theta : Z^* \to 2Z^*$ by $\theta\left(\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle\right) = \left\langle \begin{matrix} & 2a & \\ 2b & 2c & 2d \\ & 2e & \end{matrix} \right\rangle$

Clearly, θ is a homomorphism since for all $A, B \in Z^*$ we have

$\theta(A \bullet B) = \theta(A) \bullet \theta(B)$ and $\theta(A + B) = \theta(A) + \theta(B)$ belongs to $2Z^*$.

Also, θ is a bijection, since, $\theta(Z^*) = 2Z^*$.

We show that θ is one to one and onto.
Now, if $\theta : Z^* \to 2Z^*$ is define by $\theta(Z^*) = 2Z^*$.
Let $2Z^* = \{2n : n \in Z^*\}$ then $\theta(n) = 2n, \forall n \in Z^*$.
Now, if $\theta(m) = \theta(n)$. Then $2m = 2n \Rightarrow m = n$
Thus θ is one to one.
Also,

If $n \in 2Z^*$ then n is even, so $n = 2m, \forall m = \dfrac{n}{2} \in Z^*$

Hence , $\theta(m) = 2(n/2) = n$, thus θ is onto

Theorem

The ring, $O^* = \langle O^\lozenge, +, \bullet \rangle$ of all zero-heart rhotrices of size three is isomorphic to the ring $2O^* = \langle 2O^\lozenge, +, \bullet \rangle$ of all twice zero-heart rhotrices of size three.

Proof

Let the set of all zero heart rhotrices of size three be define as

$$O^\lozenge = \left(\left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle : a, b, 0, d, e \in Z \right).$$

Then it follows that

$$2O^\lozenge = \left(\left\langle \begin{matrix} & 2a & \\ 2b & 0 & 2d \\ & 2e & \end{matrix} \right\rangle : a, b, 0, d, e \in Z \right).$$

Now, since $O^* = \langle O^\diamond, +, \bullet \rangle$ and $2O^* = \langle 2O^\diamond, +, \bullet \rangle$. It is easy to show that, O^* and $2O^*$ are rings of rhotrices having the same order. We define the mapping

$\theta : O^* \to 2O^*$ by $$\theta\left(\left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle\right) = \left\langle \begin{matrix} & 2a & \\ 2b & 0 & 2d \\ & 2e & \end{matrix} \right\rangle$$

Clearly, θ is a homomorphism since for all $A, B \in O^*$ we have

$$\theta(A \bullet B) = \theta(A) \bullet \theta(B)$$

and

$$\theta(A + B) = \theta(A) + \theta(B)$$

belongs to $2O^*$.

Also, θ is a bijection, since, $\theta(O^*) = 2O^*$.

We show that θ is one to one and onto.

Now, if $\theta : O^* \to 2O^*$ is define by $\theta(O^*) = 2O^*$

Let $2O^* = \{2n : n \in O^*\}$ then $\theta(n) = 2n, \forall n \in O^*$

Now,

If $\theta(m) = \theta(n)$. Then $2m = 2n \Rightarrow m = n$

Thus θ is one to one.

Also,

If $n \in 2O^*$ then n is even, so $n = 2m, \forall m = \frac{n}{2} \in O^*$

Hence , $\theta(m) = 2(n/2) = n$, thus θ is onto

4. Conclusion

We have presented a study of certain ring; whose underlying set is the set of all real rhotrices of size three. The initial properties of rhotrix ring and along with new ones were discussed. In addition, we also, identified a number of subrings of the real rhotrix ring and developed some number of lemmas and theorems.

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