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On Certain Consequences of the Philip Hall Theorem on Finite Groups

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Abstract

Some consequences of the Philip Hall theorem were deduced, one of them is its relationship with Sylow's Theorem. Among other results we have also shown that if H and J are subgroups of G and are of the same order, then the fact that H is a Hall π-subgroup implies J is also a Hall π-subgroup.

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1. Introduction

Finite groups of different orders are being studied by many authors. The study of finite groups lead to many algebraic and combinatorial results (and problems), for example see [3, 4, 5, 8, 9, 10, 12, 13].

In 1928, Philip Hall contributed immensely in the study of finite soluble groups. He made an important discovery in the study of finite groups which may be regarded as a generalization of Sylow's Theorem for only finite soluble groups. Hall in his theorem gives the existence of certain Hall π- subgroups of a finite soluble group which arrive in a specified manner (where π is a set of primes involve in the prime factorization of the order of the group), and guarantees that these Hall π- subgroups are conjugate, it also guarantees that any π- subgroup of the group is contained in a Hall π- subgroup of the group and finally, any group satisfying these conditions must be soluble and vice versa [1]. These important results first appeared in a paper titled "*A note on soluble groups*" published by Journal of London Mathematical society see [1]. Although there are some gaps in the proof of the theorem which have been addressed by many authors for example see [2, 3]. Fairly soon after the publication of the Hall's theorem, attempts were made to fill some gaps and to reduce or to avoid complexity in the proof. In an attempt to deduce some consequences of the theorem (some of which are generalizing properties of finite soluble groups), authors like Philip Hall as in [2, 3] introduced the concept of Sylow System and Sylow bases. He also gives the proof of the converse of the theorem, which generalizes the Burnside theorem. Moreover, Hall's attempt to extend his theorem to finite insoluble groups failed. In [4], Berkovich, in the 100th anniversary of the Hall's Theorem, gives the alternative proof and some consequences of the theorem. Furthermore, Berkovich, gives easy proofs of some known results about solvable groups, see [5] for detail discussion of the proof.

However, there are certain consequences of the theorem which are yet to be discussed. In this paper, we deduce further some consequences of the theorem. Before we begin our investigations we first start with the following basic terminologies in the next section. These terminologies are standard and can be found in many group theory text books, for example see [5, 6, 7, 8, 9, 10, 11, 12, 13].

2. Preliminaries

We begin with the following well known definitions which can be found in either of [5, 6, 8, 9, 10, 11, 12, 13].

Definition 2.1 Let *G* be a finite group and *N* be a subgroup of *G* (which shall henceforth be written as $N \leq G$, i.e., its standard notation). The *index* of *N* in *G* denoted as $|G : N|$ is the number of distinct left (or right) cosets of *N* in *G*.

Definition 2.2. A subgroup *N* of a group G is said to be *normal* (or *invariant*) in *G* denoted as $N \trianglelefteq G$ if and only if $aN = Na$ for all $a \in G$. Equivalently $N \trianglelefteq G$ if and only if $ana^{-1} \in N$ for all $n \in N$, $a \in G$. A group *G* is said to be *simple* if *G* has no nontrivial normal subgroup.

Definition 2.3. The *normalizer* (or *centralizer*) of *H* (where $H \leq G$) is the set $NG(H) = \{a \in G : aHa^{-1} = H\}$. It is well known that $H \trianglelefteq NG(H) \leq G$. An element $a \in G$ is said to be a *conjugate* of $b \in G$ if there exists $x \in G$ such that $a = xbx^{-1}$ and it is said to be *self-conjugate* if $a = xax^{-1}$ for all $x \in G$. If $H, K \leq G$ then *H* is said to be conjugate of *K* if there exists $x \in G$ such that $H = xKx^{-1}$. It is well known that conjugacyin groups is an equivalence relation. We adopt the notation *Hx* to denotes xHx^{-1}. If $N \trianglelefteq G$then $G|N$ is denoted as the quotient group of *G* by *N*.

Definition 2.4. If $|G| = p^n$where *p* is a prime number and $n \in \mathrm{N}$ then *G* is said to be a *p-group*. If $|G| = p^{\alpha}m$ where $p \nmid m$ (*p* prime and $n \in \mathrm{N}$) then a subgroup of*G* of order $p^{\alpha}$is said to be a *Sylow-p-subgroup* of *G*. The number of Sylow-*p*-subgroups is denoted by n_p and the set of Sylow-*p* -subgroup is denoted by *Sylp*(*G*).

Definition 2.5. A finite sequences $G = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_n = \{e\}$ (where *e* is the identity element in *G*) of normal subgroups (resp., of subgroups) of *G* is said to be an *invariant* or *normal series* (resp. a *subnormal series* if $N_i \trianglelefteq N_{i-1}$ ($i = 1, \ldots, n$)). The factors $N_{i-1}|N_i$ ($i = 1, \ldots, n$) are called the factors of the series. A group *G* is said to be *soluble* (or *solvable*) if there exists a subnormal series of *G* with abelian factors. A subgroup *N* of a group *G* is said to be *maximal* in *G* if *N* is not contained in any other proper subgroup of *G*. A non-trivial normal subgroup of *G* (i. e., $H \trianglelefteq G$ such that $H \neq \{e\}$ and $H \neq G$) say *H* is said to be *minimal normal* in *G* if *H* does not contain a non-trivial normal subgroup of *G:* Equivalently, *H* is a minimal normal subgroup of *G* if (i) $\{e\} \leq H \trianglelefteq G$ and (ii) if $\{e\} \leq N \leq H$ and $N \trianglelefteq G$ then either $N = \{e\}$ or $N = H$.

Definition 2.6. Let *G* be a finite group and π be a set of prime numbers. A subgroup *M* of *G* is said to be a *π-subgroup* if its order is a product involving only the elements in π, and it is said to be a *Hall π-subgroup* if it is a π subgroup and its index is a product involving only primes not in π. A group G is said to be an *elementary abelian group* if every non trivial element has order a prime say *p*. Equivalently, *G* is said to be elementary abelian if $x^p = e$ for all $x \neq e \in G$, where *p* is prime. For proper understanding of the concept of finite group theory, we refer the reader to [5, 6, 9, 10, 11, 12, 13].

Next, we record some well-known results which shall be needed for our subsequent discussions.

Lemma 2.1 ([12], Sec.3, Thrm 3.38)**.** *Let G be a group,* $H \leq G$ *and* $K \trianglelefteq G$*, then* $HK \leq G$*,* $H \cap K \trianglelefteq H$ *and* $K \trianglelefteq HK$*.*

Theorem 2.2 ([14], Sylow's Thrm)**.** *Let G be a finite group of order* $p^{\alpha}m$*, where p is prime and* $p \nmid m$*. Then*

(i) Sylp(G) $\neq \{ \}$ *, i. e., Sylow-p-subgroup(s) of G exists; (ii) any two Sylow-p-subgroup of G are conjugate; (iii) the number of Sylow-p-subgroup written as np is of the form* $(1 + kp)$*, i. e.,* $n_p \equiv 1(mod\ p)$*, further,* n_p *is the index in G of the normalizer NG(p) for any Sylow-p- subgroup, thus* n_p *divides m.*

Lemma 2.3 ([6])**.** *Let G be a soluble group and* $K \leq G$*. Then,*

(i) K is soluble;

(ii) if $K \trianglelefteq G$ *then G/K is soluble.*

Lemma 2.4 ([6])**.** *If* $K \trianglelefteq G$ *such that G/K and K are soluble then G is soluble.*

Next, we state the Philip Hall theorem. This theorem has been given in many forms by different authors, although the statements are equivalent. However, for simplicity we choose to use a similar statement of the theorem as in [2] which has clear elaborations.

Theorem 2.5 ([2], Philip Hall theorem)**.** *Let G be a finite soluble group and π be set of prime numbers, then*

(i) G has a Hall π-subgroup;

(ii) any two Hall π-subgroups of G are conjugate;

(iii) any π-subgroup of G is contained in a Hall π-subgroup.

Remark 2.6. *In general for insoluble groups, Hall π-subgroups may or may not exists for certain π, as we discuss in the following example.*

Consider the alternating group of degree five denoted as A_5 having 60 elements, i. e., $|A_5| = 2^2 . 3.5$. Let π be a set of prime(s) from the factorization of order of A_5. Thus, $\pi = \{2\}$, $\pi = \{3\}$, $\pi = \{5\}$, $\pi = \{2, 3\}$, $\pi = \{2, 5\}$, $\pi = \{3, 5\}$ or $\pi = \{2, 3, 5\}$. Now, take $\pi = \{2\}$ and let $H \leq G$ such that $|H| = 2^2$. Then $|A_5 : H| = 15$ and since $3, 5 \notin \pi$, it means that by Theorem 2.5 A_5 has a Hall {2}-subgroup of order 4.

Next, take $\pi = \{3\}$ and let $H \leq G$ such that $|H| = 3$. Then $|A_5 : H| = 20$ and since $2, 5 \notin \pi$, it means that by Theorem 2.5, A_5 has a Hall {3}-subgroup of order 3.

Further, take π = {5} and let $H \leq G$ such that |*H*|= 5. Then $|A_5 : H|$= 12 and since 2, 3 $\notin \pi$, it means that by Theorem 2.5, A_5 has a Hall {5}-subgroup of order 5.
Similarly, if we take π= {2, 3} and let $H \leq G$ such that |*H*| = $2^2.3$. Then $|A_5 : H|$ = 5 and since $5 \notin \pi$, it means that by Theorem 2.5, A_5 has a Hall {2, 3}-subgroup of order 12, i. e., A_4 (the alternating group of degree 4). Thus this means that A_4 is a Hall {2, 3}-subgroup of A_5.
Moreover, take π = {2, 5} and let $H \leq G$ such that |*H*| = $2^2.5$. Then $|A_5 : H|$ = 3 and since $3 \notin \pi$, it means that by Theorem 2.5, A_5 has a Hall {2, 5}-subgroup of order 20. Furthermore, by taking π = {3, 5} and letting $H \leq G$ such that |*H*| = 3.5. Then $|A_5 : H|$ = 4 and since $2 \notin \pi$, it means that by Theorem 2.5, A_5 has a Hall {3, 5}-subgroup of order 15.
However, If we let A_5 act on the cosets of H (where H is either a Hall {2, 3}-or a Hall {3, 5}-subgroup of A_5) we can obtain a homomorphism $\varphi : A_5 \rightarrow S_t$ (where *t* = 2 or 3). Notice that $Ker\varphi \leq H$, it follows that $Ker\,\varphi \neq \{e\}$ and $Ker\,\varphi \neq A_5$. Moreover, notice that $S_2 \neq \{e\}$ and $S_3 \neq \{e\}$ which ensures that this action is nontrivial. It is well known that kernel of a homomorphism is a normal subgroup, this contradict the fact that A_5 is simple. Hence A_5 does not contain any Hall {2,3}-or a Hall {3,5}-subgroup which verified the claim.

3. Some Consequences of Philip Hall Theorem

In this section we deduce certain consequences of Philip Hall theorem. One of the consequences is its relationship with the Sylow's theorem. We first begin with the following theorem.

Theorem 3.1. *Let G be a finite group and H be a Hall π-subgroup of G then,*
(i) If J ≤ G with |J| = |H| then J is a Hall π-subgroup of G and is conjugate to H;
(ii) for every element g∈G, Hg is a Hall π-subgroup of G;
(iii) If H ≤ L ≤G then H is a Hall π-subgroup of L.
Proof. Let *G* be a finite group and *H* be a Hall π-subgroup of *G*.
(i) Let $J \leq G$ with |*J*| = |*H*|. Now since |*J*| = |*H*| then *J* is a Hall π-subgroup of *G* and thus by Theorem 2.5 (ii), *J* and *H* are conjugate.
(ii) Let $g \in G$. Notice that *H* is a Hall π-subgroup of *G*, thus *Hg* = *H*and therefore |*Hg*|= |*H*|. Therefore by Theorem 2.5 (ii) *Hg* and *H* are conjugate and hence *Hg* is a Hall π-subgroup of *G*, as required.
(iii) Let $H \leq L \leq G$. If *H* = *L* the result follows. Now suppose $H \leq L$ then since *H* is a Hall π-subgroup of *G*, the result follows from Theorem 3.1 (i).
Now we have the next consequence.

Theorem 3.2. *Let H ≤ G and K ⊴ G. Then*
(i) If H is a Hall π-subgroup of G then HK \ K is a Hall π-subgroup of G \ K;
(ii) If H is a Hall π-subgroup of G then H ∩ K is a Hall π-subgroup of K;
(iii) If H is a π-subgroup of G then HK \ K is a π-subgroup of G \ K.
Proof. Let *G* be a group, *H ≤ G and K ⊴ G.*
(i) Let *H* be a Hall π-subgroup of *G*. Notice that *K* is normal in *G*, thus *G* \ *K* is a group and by

Lemma 2.1 $HK \leq G$ which implies that HK is soluble and possess a Hall π-subgroup. Notice that $HK \setminus K$ is a group which implies that $HK \setminus K \leq G \setminus K$. Thus by Theorem 2.5 $HK \setminus K$ is soluble and thus a Hall π-subgroup of $G \setminus K$ as required.

(ii) Let H be a Hall π-subgroup of G. Notice that K is normal in G, thus $H \cap K \leq K$ and $H \cap K \trianglelefteq H$. This means that $H \cap K$ is soluble and possess a Hall π-subgroup of G. Hence by Theorem 3.1 $H \cap K$ is a Hall π-subgroup of K.

(iii) This follows from (i).

Remark 3.3. *Notice also that equivalently the proof of Theorem 3.2 follows from Theorem 1.2.6 in [4].*

Theorem 3.4. *Let G be a soluble group of order $p^\alpha q^\beta$. Then G have a Hall π-subgroup of order p^α and Hall π-subgroup of order q^β.*

Proof. For the former, take $\pi = \{p\}$ and $\alpha \in$N. The result follows from Theorem 2.5 (i). Similarly, for the later.

The next result is the relationship between Philip Hall theorem and Sylow's Theorem on soluble groups.

Theorem 3.5. *Let G be a finite soluble group of order $p^\alpha m$ where p is prime and α* ∈N. *Then,*
(i) G has a Hall π-subgroup of order p^α if and only if G has a Sylow-p-subgroup of order p^α;
(ii) any two Hall π-subgroup of order p^α are conjugate in G if and only if any two Sylow-p-subgroups of order p^α are conjugate in G.

Proof. Let G be a finite soluble group of order $p^\alpha m$, where p is prime and $\alpha \in$N.

(i) Notice that G is soluble. Thus by Theorem 2.5 G has a Hall π-subgroup say H of order p^α, i. e., $|H| = p^\alpha$and $p \nmid |G : H|$ then $p \nmid m$. Thus by Sylow's Theorem there exists a Sylow-p-subgroup of order p^α.

Conversely, suppose G has a Sylow-p-subgroup of order p^α say H. Then $p \nmid m$. Let $\pi = \{p\}$, notice that $|G : H|= m$ and $m \notin \pi$ thus by Theorem 3.4 H is a Hall π-subgroup of order p^α.

(ii) Suppose any two Hall π-subgroup say H and K of order p^αareconjugate in G. Then by (i) H and K are Sylow-p-subgroup of G. Notice that H and K have the same order and thus by Sylow's Theorem H and K are conjugate in G.

Conversely, suppose any two Sylow-p-subgroup say H and K of order $p^\alpha$are conjugate in G. Thus by (i) H and K are Hall π-subgroup of G. Notice that H and K are having the same order then by Theorem 2.5 (ii) H and K are conjugate. This complete the proof.

Example 3.6. Consider the alternating group of degree 4, i. e., A_4, having order 12. Thus, $|A_4| = 2^2.3$. It is easy to see that by Theorem 2.2 that A_4 has a Sylow-p-subgroup of order 2^2 and any two Sylow-p-subgroups of order 4 are conjugate. Now take π as $\{2\}$ and let H be any Sylow-p-subgroup of A_4 of order 4. Notice that $|A_4 : H| = 3$ and notice also that $3 \notin \pi$. Thus, by definition H is a Hall-$\{2\}$-subgroup of A_4 of order 4 and vice versa.

4. Conclusion

We have successfully reviewed and demonstrated how we deduced certain consequences of the Philip Hall Theorem and have shown how it is related to the Sylow's Theorem. Further consequences can also be derived from the theorem which should further give elaborations on finite soluble groups and as well help in categorizing them.

References

[1] Hall, P. A note on soluble groups. *Proc. Lond. Math. Soc.* **2** (1928), 98-105.

[2] Hall, P. A characteristic property of soluble groups. *Proc. Lond. Math. Soc.* **12** (1937), 198-200.

[3] Hall, P. On the Sylow system of a soluble group. *Proc. Lond. Math. Soc.* **43** (1937), 316-323.

[4] Berkovich, Y. Alternate proof of two theorems of Philip Hall on p- groups and some related results. *J. Algebra.* **294** (2005), 463-477.

[5] Berkovich, Y. Alternate proof of the two classical theorems on finite solvable groups and some related results for p-Groups. *Glasnik Matematicki*. **45** (2010), 431-439.

[6] Burnside, W. *Theory of groups of finite order.* Cambridge University Press, (1879).

[7] David, S. D. and Richard, M. F. *Abstract algebra* John Wiley and Sons, Inc., (2004).

[8] Green, J. A., Roseblade, J. E. and Thompson, J. G. FRS. Philip Hall. *Proc. Lond. Math. Soc.* **16** (1983), 603-626.

[9] Hartley, B. Collected work of Philip Hall. *Proc. Lond. Math. Soc.* **25** (1993), 89-90.

[10] Herstein, I. N. *Topics in Algebra*. Second Edition, New York, John Willey and Sons,(1964).

[11] Herstein, I. N. *Topics in Algebra*. Third edition, New York, John Willey and Sons, (2003).

[12] John, S. R. *A course on group theory*. Cambridge University Press, (1978).

[13] Kuku, A. O. *Abstract algebra.* Ibadan University Press, (1980).

[14] Singh, G. and Dar, A. M. The Sylow theorem and its consequences. *Math. Theory and Modeling* **4** (2014), 185-190.