



A Priori Estimates for Navier-Stokes Equations in a Strip: A Case With Nonzero Flux Condition in Uniformly Local Spaces

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Abstract

This paper computes energy estimates for Navier-Stokes equations in a strip in uniformly local spaces; focus is on a case with nonzero flux condition. In view of the instability properties of the poiseuille flow at certain critical lengths, a special auxiliary function $u(x_2)$ of poiseuille solution with more stable properties is constructed and then subtracted from the solution $u(x)$ of the Navier-Stokes equations. The resulting system, which also has an extra term to be controlled, is then treated as the case of zero flux condition. This article specifies criteria for the choice of the poiseuille-like solution and provide bounds for it. Next, apriori estimates and bounds for the solutions of the entire system is computed in uniformly local spaces.

Keywords: Navier-Stokes equations, Poiseuille flow, uniformly local spaces, weighted energy spaces, zero flux

1. Introduction

The mathematical analysis of Navier-Stokes equations started at the beginning of the 20th century by Leray, 1933. His work focused on the correct formulations of initial boundary problems for Navier-Stokes equations; proofs of existence and uniqueness of solutions in different functional spaces, investigation of solutions' regularity, construction of asymptotics, etc. Most studies on Navier-Stokes consider bounded underlying domain with zero flux condition or the use of finite energy solution in the whole space $\mathbb{R}^2$; as yet, not much is known about infinite energy solutions. The method used for the investigation of this equation is the so-called *energy estimates*. When dealing with unbounded domain with general flux condition as the case to be considered here, it is much more involved than the bounded case with zero flux.

Anthony and Zelik, 2014, studied Navier-Stokes equations in an infinite channel with solutions obtained in uniformly local spaces. Existence, uniqueness and further regularity results were established with more focus on zero flux condition. Further to this work, we want to emphasize aspects of this work with a more general case of nonzero flux which has more physical relevance.

This paper is organized as follows: Section 2 provides the basic function spaces for the treatment of Navier-Stokes Equations and provides insight on weighted energy spaces and their relationship with uniformly local spaces. Section 3 introduces our model and describes the problem to be solved in this paper; in section 4, we subtract a special stable poiseuille solution from the Navier Stokes equation. Section 5 establishes bounds for the extra term of the Navier-Stokes equation occasioned by the introduction of the stable Poiseuille like solution. In section 6, we close our estimates for the entire Navier Stokes system with general flux condition.

2. Preliminaries

2.1. Function Spaces for Navier-Stokes Equations

We state here standard mathematical spaces for Navier-Stokes equations as given by Temam, 1988; Ladyzhenskaya, 1972; Ladyzhenskaya, 1969; Babin, 1992. The basic functional spaces is the Lebesgue space $L^2(\Omega)$ with scalar product $(u, v) = \sum_j \int u_j(x)v_j(x)dx$ and norm $|\cdot| = (\cdot, \cdot)^{\frac{1}{2}}$. We will also need the Sobolev space $\mathbb{H}^{k,p}(\Omega)$ where $k \in \mathbb{N}$ and $p \in [0, \infty)$ consisting of all $u \in L^p(\Omega)$ whose weak derivatives up to order k is in $L^p(\Omega)$. Moreover, $\mathbb{H}^{k,p}(\Omega)$ is a separable Hilbert space with norm:

$$\|u\|_{\mathbb{H}^{k,p}(\Omega)} \leq \left(\sum_{|\alpha| \leq k} \int |D^\alpha u(x)|^p dx \right)^{\frac{1}{p}}$$

Obviously, $\mathbb{H}^{k,2}(\Omega)$ is a Hilbert space where $k \in \mathbb{N}$. The weak solution to the Navier-Stokes problem is needed; for this, a proper space of test functions is also required. Now, take $V = V(\Omega) = \{\phi \in C_0^\infty(\Omega) : \text{div} \phi = 0 \text{ in } \Omega\}$

The closure in $L^2(\Omega)$ and in $\mathbb{H}^{1,2}(\Omega)$ is denoted by H and V respectively. The scalar product norms in those two spaces are those inherited from $L^2(\Omega)$ $\mathbb{H}^{1,2}(\Omega)$ respectively.

The domain Ω is a Poincare domain if there exist $\lambda_1 > 0$ such that

$$\int \phi^2 dx \leq \frac{1}{\lambda_1} \int |\nabla \phi|^2 dx, \phi \in C_0^\infty(\Omega) \quad (1)$$

The inequality (1) is called the Poincare inequality; it can be shown that if Ω is bounded in some direction, i.e. there is a vector $\mathbf{b} \in \mathbb{R}^2$ such that $\sup |(x, \mathbf{b})| < \infty$ then Ω is a Poincare domain.

If Ω is a Poincare domain, then the original norm on V is equivalent to the norm $\|\cdot\|$ induced by the scalar product

$$((u, v)) = \int \sum_{j=1}^2 \nabla u_j \cdot \nabla v_j dx = (\nabla u, \nabla v), u, v \in V \quad (2)$$

We define next a bilinear form $a: V \times V \rightarrow \mathbb{R}$ by

$$a(u, v) = (\nabla u, \nabla v), u, v \in V \quad (3)$$

Since obviously the form a coincides with $((\cdot, \cdot))$ scalar product in V , it is V – continuous i.e. $|a(u, v)| \leq C \|u\|^2$ for some $C > 0$ and $u \in V$. Hence, by Riesz Lemma, there exists a unique linear operator $A: V \rightarrow V'$ where V' is the dual of V such that $a(u, v) = (Au, v)$ $u, v \in V$. Moreover, the form a is obviously V – coercive i.e. it satisfies $a(u, v) \geq \alpha \|u\|^2$ for some $\alpha > 0$ and $u \in V$. Therefore, by means of Lax-Milgram theorem (see Brzezniak, 1991 Theorem II 2.1) the operator $A: V \rightarrow V'$ is an isomorphism. Since, V is densely and continuously embedded into H and H can be identified with its dual H' , we have the following embeddings

$$V \subset H \cong V' \subset H' \quad (4)$$

Let us then recall that we say that the spaces V, H and V' form a Gelfand triple. Next we define an unbounded linear operator A in H as follows

$$\begin{cases} D(A) = \{u \in V, Au \in H\} \\ Au = Au, u \in D(A) \end{cases} \quad (5)$$

It is now well established that under some additional assumptions related to the regularity of the domain D , the $D(A)$ can be characterized in terms of Sobolev spaces.

We introduce and briefly discuss the weighted and uniformly local spaces which are the main technical tools to deal with infinite-energy solutions, see Babin, 1992; Effendiev and Zelik, 2002 for more detailed exposition. These tools will help us to obtain estimates for our equations (2) in unbounded domain $\Omega = \mathbb{R} \times (-1, 1)$. We explain the space as follows: let us define $B_{x_0}^1$ –a unit rectangle centered at $(x_0, 0)$ represented as:

$$B_{x_0}^1 = \left(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right) \times (-1, 1), x_0 \in \mathbb{R} \quad (6)$$

Definition 1. A function $\phi \in C_{loc}(\mathbb{R})$ is weight function of exponential growth rate $\mu > 0$ if the following inequalities hold:

$$\phi(x + y) \leq C_\phi \phi(x) e^{\mu|y|}, \quad \phi(x) > 0$$

For all $x, y \in \mathbb{R}$. See Zelik, 2008.

The next result, from Anthony and Zelik, 2014 establishes the relationship between weighted energy spaces and uniformly local spaces to be used in this paper.

Lemma 1. Let ϕ be a weight function of exponential growth rate, where $\phi_{x_0}(x) = \phi(x - x_0)$, satisfying $\int \phi^2 dx < \infty$ then the following hold

$$\|u\|_{L_\phi^2} \leq C_1 \|u\|_{L_b^2}^2 \cdot \int \phi^2 dx \quad (7)$$

$$\|u\|_{L_b^2}^2 \leq C_2 \sup_{x_0 \in \mathbb{R}} \left(\|u\|_{L_{\phi_{x_0}}^2}^2 \right) \quad (8)$$

where C_1 and C_2 depend only on C_ϕ and μ from **definition 1**. Also see Zelik, 2003; Zelik, 2008; Zelik, 2007; Zelik, 2013 for more insight.

3. A Brief Description of the Problem

We now consider the general case of nonzero flux condition for the following system:

$$\begin{cases} \partial_t u + (u, \nabla)u - \Delta u + \nabla p = g \\ \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \end{cases} \quad (9)$$

where, $\partial_t u$ is the partial derivative of the velocity vector with respect to t , $(u, \nabla)u$ is the nonlinearity, Δu is the viscous term and ∇p is the pressure gradient, g is the external force.

In general, the problem possesses the mean flux first integral:

$$Su_1 = \frac{1}{2} \int_{-1}^1 u_1(t, x_1, s) ds = c, \quad (10)$$

In a strip $\Omega = \mathbb{R} \times [-1, 1]$. Where the constant c may depend on t ($c = c(t)$), but is independent of x_1 . The case of zero flux is a bit more transparent:

$$c = 0 \quad (11)$$

Generally speaking, the Navier Stokes Equation (9) with $g = 0$ possesses the classical poiseuille solution

$$u_c(x) = \left(\frac{3}{2}c(1 - x_2^2), 0\right) \quad (12)$$

If (10) is satisfied, then the difference $u - u_c$ has zero flux. Therefore, we may easily define the weak solution of (9) as a function of $w = u - u_c$ in the sense of distribution over the divergent free vector fields.

Remark: We present here a succinct description of velocity profile of a Poiseuille flow as given by Doering, 2004: At certain length, the Poiseuille flow theory always provides an accurate description of flow. At higher lengths, the theory applies only after some distance down the strip; lower lengths are so short that they could be ignored. But as the lengths increase, the situation becomes more interesting. Particularly, the case in which the fluid enters with uniform speed over the whole cross-section. Because of no-slip condition, the fluid next to the wall must immediately be slowed down. This retardation spreads inwards while the fluid at the centre must move faster, so the average speed remains the same while the mass is conserved. One thus get sequences of velocity profiles.

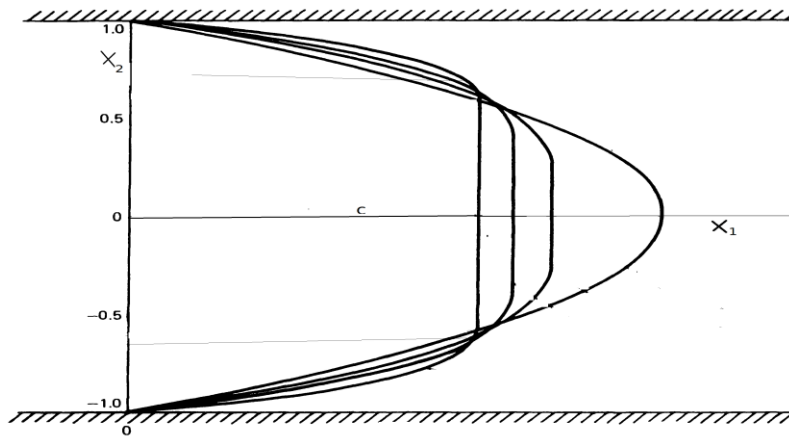


Figure 1: Velocity Flow profile in a Strip

4. Subtraction of special stable Poiseuille solution $\sim u(x_2)$ from NS solution $u(x)$

In view of the above instability properties cited, we construct a special auxiliary function $\tilde{u}(x_2)$ of poiseuille solution exhibiting a more stable property.

Let (10) be satisfied, then, $S(u_1 - u_c) \equiv 0$ therefore, we may define a weak solution of u by substituting $\tilde{w} = u - u_c$ in (1) with $S\tilde{w} \equiv 0$ and treat as the well known case of zero flux. However, we are confronted with instability nature of poiseuille solution which may create a setback in the circumstances; hence, we will instead construct a special solution of the Navier Stokes Equation (9)-(10) of the form:

$$\tilde{u}(x) = (\tilde{u}(x_2), 0), \tilde{u}(\pm 1) = 0, \quad (13)$$

Such that

$$S\tilde{u} \equiv c; \quad (14)$$

with the proportionate non-zero external force S

$\tilde{g} = 0$. We introduce now the new unknown

$$w = u - \tilde{u} \quad (15)$$

where w is the solution of the NS equation satisfying the zero flux condition and u satisfies the constant flux condition. We substitute (15) into (9):

$$\partial_t w + \partial_t \tilde{u} + (w + \tilde{u}) \cdot \nabla (w + \tilde{u}) + \nabla p = \Delta(w + \tilde{u}) + \tilde{g}$$

which reduces to:

$$\partial_t w + \partial_t \tilde{u} + \tilde{u} \cdot \nabla w + w \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + w \cdot \nabla w + \nabla p - \nu \Delta w - \nu \Delta \tilde{u} = \tilde{g}. \quad (16)$$

The second and the fifth terms of the LHS of (16) disappear because of the choice of $\tilde{u}$. Therefore, w solves the following:

$$\begin{cases} \partial_t w + (\tilde{u}, \nabla) w + (w, \nabla) \tilde{u} - \Delta w - \Delta \tilde{u} + (w, \nabla) w + \nabla p = \tilde{g} \\ \operatorname{div} w = 0, \quad w|_{\partial\Omega} = 0, \quad w|_{t=0} = w_0 = u_0 - \tilde{u}_0 \end{cases} \quad (17)$$

$$Sw_1 \equiv 0 \quad (18)$$

which differs from (9) by an additional nonlinear operator and the laplacian terms respectively

$$(\tilde{u}, \nabla) w + (w, \nabla) \tilde{u} = L_{\tilde{u}} w, \text{ and } \Delta \tilde{u} \quad (19)$$

The next lemma specifies the choice of $\tilde{u}(x_2)$ and its corresponding bounds in the extra terms (19).

Lemma 2. Let $c \in \mathbb{R}$ be arbitrary. Then, there is a vector field $\tilde{u}(x) = ((\tilde{u}(x_2), 0), \tilde{u}(\pm 1)) = 0$ such that

$$(L_{\tilde{u}} w, \varphi^2 w) \leq \frac{1}{2} \|w\|_{W_\varphi^{1,2}(\Omega)}^2 \quad \forall w \in W_0^{1,2}(\Omega); \quad (20)$$

and

$$\|\tilde{u}\|_{C([-1,1])} \leq k|c| \|\tilde{u}'\|_{L^2([-1,1])} \leq k|c|^{3/2}, \quad (21)$$

$$\|\tilde{u}''\|_{L^2[-1,1]} \leq k|c|^{3/2} \quad (22)$$

where the constant k is independent of c and $\tilde{g}$

Proof:

We refer to our special stable poiseuille-like solution, $\tilde{u}(x_2)$. Based on our choice of this function, it satisfies the equation of the parabola passing through the points: $A(-1,0)$, $B(-1 + \delta, \lambda)$, $C(1 - \delta, \lambda)$, $D(1,0)$, λ is close to c and $\delta \ll 1$.

Moreover, parabola ABCD satisfies the equations:

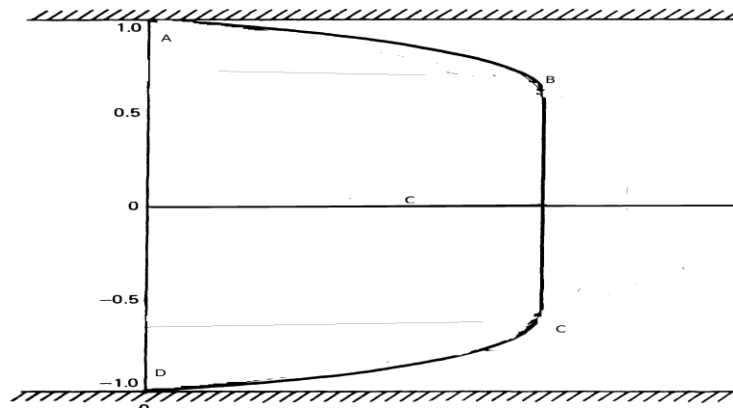


Figure 2: Velocity Flow profile of $u(x_2)$ in a Strip

$$\tilde{u}(x_2) = \begin{cases} AB: \lambda - \frac{\lambda}{\delta^2} (x_2 + 1 - \delta)^2, & x_2 \in [-1, -1 + \delta] \\ BC: \lambda, & x_2 \in [-1 + \delta, 1 - \delta] \\ CD: \lambda - \frac{\lambda}{\delta^2} (x_2 - 1 + \delta)^2 & x_2 \in [1 - \delta, 1] \end{cases} \quad (23)$$

where the two parameters $\lambda \sim c, \delta \sim c^{-1}$ are chosen in such way that (14) is satisfied. Then the straightforward calculations show that the other assumptions of the lemma are also satisfied, see Zelik, 2007 for more details.

In order to satisfy the flux condition, we compute the integral under the curve:

$$\int_{-1}^1 \tilde{u}(x_2) dx_2 = \int_{-1}^{-1+\delta} \tilde{u}(x_2) dx_2 + \int_{-1+\delta}^{1-\delta} \tilde{u}(x_2) dx_2 + \int_{1-\delta}^1 \tilde{u}(x_2) dx_2 = \frac{4\lambda\delta}{3} + (2 - 2\delta)\lambda \quad (24)$$

So, we need to fix δ in such a way that (20) is satisfied. Let $w \in \{W_0^{1,2}(\Omega)\}^2$, then, by direct calculations on $L_{\tilde{u}}w$, we get:

$$(w, \nabla) \tilde{u} = \sum_{i,j=1}^2 w_i \partial_{x_i} \tilde{u}_j = (w_1 \partial_{x_1} + w_2 \partial_{x_2}) \begin{pmatrix} \tilde{u}_1(x_2) \\ 0 \end{pmatrix}. \quad (25)$$

Since $\tilde{u}_1(x_2)$ has derivative only in the x_2 direction and $\tilde{u}_2(x_1, x_2) = 0$; therefore:

$$(w, \nabla) \tilde{u} = \sum_{i,j=1}^2 w_i \partial_{x_i} \tilde{u}_j = \begin{pmatrix} w_2 \partial_{x_2} \tilde{u}_1 \\ 0 \end{pmatrix}; \quad (26)$$

the second term of the nonlinearity reduced as:

$$\sum_{i,j=1}^2 \tilde{u}_i \partial_{x_i} w_j = \begin{pmatrix} \tilde{u}_1 \partial_{x_1} w_1 + \tilde{u}_2 \partial_{x_2} w_1 \\ \tilde{u}_1 \partial_{x_1} w_2 + \tilde{u}_2 \partial_{x_2} w_2 \end{pmatrix} = \begin{pmatrix} \tilde{u}_1 \partial_{x_1} w_1 \\ \tilde{u}_1 \partial_{x_1} w_2 \end{pmatrix} \quad (27)$$

We substitute the RHS of (25) and (27) into (17), then, multiply by $\varphi^2 w - v_\varphi$ and integrate over the domain.

In this Lemma, we are interested in the term associated with the nonlinearity for multiplication by $\varphi^2 w$ only; the rest term is considered later.

$$(L_{\tilde{u}}w, \varphi^2 w) = \left(\begin{pmatrix} w_2 \partial_{x_2} \tilde{u}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi^2 w_1 \\ \varphi^2 w_2 \end{pmatrix} \right) + \left(\begin{pmatrix} \tilde{u}_1 \partial_{x_1} w_1 \\ \tilde{u}_1 \partial_{x_1} w_2 \end{pmatrix}, \begin{pmatrix} \varphi^2 w_1 \\ \varphi^2 w_2 \end{pmatrix} \right) = (\partial_{x_2} \tilde{u}_1, \varphi^2 w_1 w_2) + C\varepsilon(|w|^2, \tilde{u}_1). \quad (28)$$

If we maintain that the second term of (28) is just $\varepsilon - small$, we may write:

$$(L_{\tilde{u}}w, \varphi^2 w) = (\partial_{x_2} \tilde{u}_1, \varphi^2 w_1 w_2) = (\tilde{u}'^{1/2} \varphi w_1, \tilde{u}'^{1/2} \varphi w_2). \quad (29)$$

By Holder's inequality we obtain:

$$(\tilde{u}'^{1/2} \varphi w_1, \tilde{u}'^{1/2} \varphi w_2) \leq \frac{1}{2} (|\tilde{u}'|, \varphi^2 w_2^2). \quad (30)$$

From (23), $|\tilde{u}'|_{[-1, -1+\delta]} = \frac{2\lambda}{\delta^2} (-x_2 + (-1 + \delta))$, $|\tilde{u}'|_{[1-\delta, 1]} = \frac{2\lambda}{\delta^2} (-x_2 - 1 + \delta)$,

and $|\tilde{u}'|_{[-1+\delta, 1-\delta]} = 0$.

Substituting the RHS of (30) and assume that $-x_2 + (-1 + \delta) < \delta$, we obtain,

$$\frac{2\lambda}{\delta^2} \int_{-1}^1 (-x_2 - 1 + \delta) \varphi^2 w^2 dx_2 \leq \frac{\lambda}{\delta} \left(\int_{-1}^{-1+\delta} \varphi^2 w^2 dx_2 + \int_{1-\delta}^1 \varphi^2 w^2 dx_2 \right). \quad (31)$$

Making the following change of variable:

$$Z(x_2) = \varphi w, Z(\pm 1) = 0;$$

From the second term of RHS of (31) we obtain:

$$\frac{\lambda}{\delta} \int_{-1}^{-1+\delta} \varphi^2 w^2 dx_2 = \frac{\lambda}{\delta} \int_{1-\delta}^1 \left(\int_{x_2}^1 Z(x_2) dx_2 \right)^2 \quad (32)$$

By Holder's inequality, we obtain from RHS (32)

$$\frac{\lambda}{\delta} \int_{1-\delta}^1 \left(\int_{x_2}^1 Z(x_2) dx_2 \right)^2 \leq \frac{\lambda}{\delta} \int_{1-\delta}^1 \delta \|Z'\|_{L^2[1-\delta,1]}^2 dx_2 = \delta \lambda \|Z'\|_{L^2[1-\delta,1]}^2. \quad (33)$$

Similarly, the first term of the RHS of (31) yields $\delta \lambda \|Z'\|_{L^2[1-\delta,1]}^2$ which reduced (22) to:

$$2\delta \lambda \int_{-1}^1 \|Z'\|_{L^2[-1,1]}^2 dx_2 = 2\delta \lambda \|w\|_{W_\varphi^{1,2}}^2, \quad (34)$$

hence, $(L_{\tilde{u}} w, \varphi^2 w) \leq 2\delta \lambda \|w\|_{W_\varphi^{1,2}}^2$,

and we only need to satisfy the following conditions:

$$2c = 2\lambda - \frac{2}{3}\lambda\delta, \quad 0 < \delta \leq 1, \quad \lambda\delta \leq \frac{1}{4}. \quad (35)$$

These inequalities would be satisfied if we take e.g. $\delta = \min\left\{1, \frac{1}{4|\lambda|}\right\}$. We have,

$$\|\tilde{u}\|_{C([-1,1])} = \sup_{x_2} |\tilde{u}(x_2)| \leq k|c|. \quad (36)$$

Next, we compute:

$$\|\tilde{u}'\|_{L^2([-1,1])} = \left(\int_{-1}^1 |\tilde{u}'|^2 dx_2 \right)^2 = \frac{8}{3} \lambda \delta^{-1/2} \sim |c|^{3/2}. \quad (37)$$

From the last inequality of (35), we posit that $\delta \leq \frac{1}{4\lambda}$ where λ is close to c . Therefore,

$$\|\tilde{u}'\|_{L^2([-1,1])} \leq k|c|^{3/2};$$

We also have that

$$\left(\int_{-1}^1 |\tilde{u}''|^2 dx_2 \right)^2 = 8c\delta^{-3/2} \sim |c|^{5/2}.$$

This completes the proof of the Lemma. The next section closes the estimate for the entire Navier-Stokes system.

5. A priori estimates for Navier-Stokes Equations with non-zero flux condition

Theorem: *Let w be sufficiently regular solution of Navier – Stokes equations 9 – 10. Then, the following estimates hold true:*

$$\|w\|_{L^\infty(0,T;L_b^2)} + \|\nabla w\|_{L^2_b(0,T;L_b^2)} \leq C \left(\|w_0\|_{L_b^2} + \|\tilde{g}\|_{L_b^2} + k \left(|c|^{\frac{3}{2}} + |c| + 1 \right)^2 \right) \quad (38)$$

Proof:

Since we now have the estimate for the additional nonlinearity $(L_{\tilde{u}} w$ from Lemma 2, we follow the same procedure for the case of zero flux condition, to find the estimates. Indeed, proving the analogue of basic a priori estimate Anthony and Zelik, 2014; we would only have the extra expression:

$$L_{\tilde{u}} w + \Delta \tilde{u} \quad (39)$$

to control.

Treating the nonlinearities $(w, \nabla) \tilde{u}$, $(\tilde{u}, \nabla) w$, and $(w, \nabla) w$ as external forces, we have the following equations when we multiply (17) by $\varphi^2 w - v_\varphi$ and integrate over the domain $\Omega = [-1,1] \times \mathbb{R}$:

$$\frac{d}{dt} \left(\frac{1}{2} \|w\|_{L_b^2}^2 - (w, v_\varphi) + (L_{\tilde{u}} w, \varphi^2 w) + (\nabla w, \nabla(\varphi^2 w)) \right) = (\tilde{g} + \Delta \tilde{u} - ((w, \nabla)w, \varphi^2 w - v_\varphi) + ((w, \nabla)\tilde{u} + (\tilde{u}, \nabla)w, v_\varphi)). \quad (40)$$

The rest of the terms in $\varphi^2 w$ and v_φ are similarly treated like those obtained for Navier-Stokes equations in Anthony and Zelik, 2014.

Next, we write estimates for the extra terms in (40) thus:

$$(L_{\tilde{u}} w, v_\varphi) = ((w, \nabla)\tilde{u}, v_\varphi) + ((\tilde{u}, \nabla)w, v_\varphi). \quad (41)$$

Considering the first term on the RHS of (41) we obtain:

$$(|\tilde{u}| \cdot \varphi |\nabla w|, \varphi^{-1} |v_\varphi|)_{L^1} = \|\tilde{u}\|_{C[-1,1]} \|\nabla w\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-1}}^2}. \quad (42)$$

Substituting (21) into (42) we obtain:

$$\|\tilde{u}\|_{C[-1,1]} \|\nabla w\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-1}}^2} \leq k|c| \|\nabla w\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-1}}^2}. \quad (43)$$

By Holder's inequality on (43) we obtain:

$$k|c| \|\nabla w\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-1}}^2} \leq k|c| \left(\frac{\epsilon \|\nabla w\|_{L_\varphi^2}^2}{2} + \frac{\|v_\varphi\|_{L_\varphi^2}^2}{2\epsilon} \right). \quad (44)$$

The second term on the RHS of (41) is estimated thus:

$$|((w, \nabla)\tilde{u}, v_\varphi)| = |((\varphi w, \varphi \nabla)\tilde{u}, \varphi^{-2} v_\varphi)| \leq \|w\|_{L_\varphi^6} \|\nabla \tilde{u}\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-2}}^3}. \quad (46)$$

By the embedding of H^1 in L^6 we obtain:

$$\|w\|_{L_\varphi^6} \|\nabla \tilde{u}\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-2}}^3} \leq \|\nabla w\|_{L_\varphi^2} \|\nabla \tilde{u}\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-2}}^3}. \quad (47)$$

And by Holder's inequality on (47) we get:

$$\|\nabla w\|_{L_\varphi^2} \|\nabla \tilde{u}\|_{L_\varphi^2} \|v_\varphi\|_{L_{\varphi^{-2}}^3} \leq \|\nabla w\|_{L_\varphi^2} \left(\frac{\epsilon \|\nabla w\|_{L_\varphi^2}^2}{2} + \frac{\|v_\varphi\|_{L_\varphi^2}^2}{2\epsilon} \right). \quad (48)$$

Since

$$\|\tilde{u}'\|_{L_\varphi^2} \leq \epsilon^{-1/2} \|\tilde{u}\|_{W_\varphi^{1,2}}, \quad (49)$$

We write

$$\epsilon^{-1/2} \|\tilde{u}\|_{W_\varphi^{1,2}} \left(\frac{\epsilon \|\nabla w\|_{L_\varphi^2}^2}{2} + \frac{\|v_\varphi\|_{L_\varphi^2}^2}{2\epsilon} \right) \leq \frac{C}{\epsilon^{1/2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \left(\frac{\epsilon \|\nabla w\|_{L_\varphi^2}^2}{2} + \frac{\|v_\varphi\|_{L_\varphi^2}^2}{2\epsilon} \right) \quad (50)$$

So, estimate for the second additional term yields:

$$|(L_{\tilde{u}} w, v_\varphi)| \leq \frac{C}{\epsilon^{1/2}} k \left(|c|^{\frac{3}{2}} + |c| \right) (\|\nabla w\|_{L_\varphi^2}^2 + C \sup_t \|w(t)\|_{L_\varphi^2}^2) \quad (51)$$

$$|(L_{\tilde{u}} w, \varphi^2 w - v_\varphi)| \leq \frac{\|w\|_{W_\varphi^{1,2}}^2}{2} \frac{C}{\epsilon^{1/2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \left(\frac{\epsilon \|\nabla w\|_{L_\varphi^2}^2}{2} + \frac{\|v_\varphi\|_{L_\varphi^2}^2}{2\epsilon} \right) \quad (52)$$

Where the constant k is independent of ϵ , v_φ , c , and w .

Next, we estimate the term:

$$|(\Delta \tilde{u}, \varphi^2 w - v_\varphi)| = k(|c|^{\frac{5}{2}} + |c|)^2 + C\epsilon \sup_t \|w(t)\|_{L_\varphi^2}^2 + \frac{C}{\epsilon} k(|c|^{\frac{3}{2}} + |c|)^{21-} + C\|\nabla w\|_{L_\varphi^2}^2. \quad (53)$$

We recall (17) and multiply by $\varphi^2 w - v_\varphi$ over the domain; the pressure term disappears because $\text{div}(\varphi^2 w - v_\varphi) = 0$. The following estimates were obtained:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|w\|_{L_\varphi^2}^2 - (w, v_\varphi) \right) + C\epsilon \|w\|_{L_\varphi^2} \|\nabla w\|_{L_\varphi^2}^2 - C\|\nabla w\|_{L_\varphi^2}^2 \left(\epsilon \sup_{t \in [0, T]} \|w(t)\|_{L_\varphi^2} \right) + C\|\nabla w\|_{L_\varphi^2}^2 - \\ & \frac{\|\nabla w\|_{L_\varphi^2}^2}{2} - \frac{\epsilon \|w\|_{L_\varphi^2}^2}{2} \leq C\epsilon^{\frac{1}{2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \left(\|\nabla w\|_{L_\varphi^2}^2 + C \sup_{t \in [0, T]} \|w(t)\|_{L_\varphi^2}^2 \right) + \frac{1}{2} \|w\|_{W_\varphi^{1,2}}^2 + \\ & k(|c|^{\frac{5}{2}} + |c|)^2 + C\epsilon \sup_t \|w(t)\|_{L_\varphi^2}^2 + \frac{C}{\epsilon} k(|c|^{\frac{3}{2}} + |c|)^2 + \frac{\|\nabla w\|_{L_\varphi^2}^2}{2} + \epsilon \|w\|_{L_\varphi^2}^2 + C\|\tilde{g}\|_{L_\varphi^2}^2 + \frac{\|w\|_{L_\varphi^2}^2}{2} + \\ & \epsilon^2 \sup_t \|w(t)\|_{L_\varphi^2}^2. \end{aligned} \quad (54)$$

Absorbing similar terms and simplifying, we get:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|w\|_{L_\varphi^2}^2 - (w, v_\varphi) \right) + C\|\nabla w\|_{L_\varphi^2}^2 \left(1 - \epsilon \sup_{t \in [0, T]} \|w(t)\|_{L_\varphi^2} \right) + \gamma \|w\|_{L_\varphi^2}^2 \leq \frac{C}{\epsilon} k(|c|^{\frac{3}{2}} + |c|)^2 + \\ & k(|c|^{\frac{5}{2}} + |c|)^2 + C\|\tilde{g}\|_{L_\varphi^2}^2 + C\epsilon^{\frac{1}{2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \sup_t \|w(t)\|_{L_\varphi^2}^2. \end{aligned} \quad (55)$$

In order to prove the key a priori estimates (38), we state the following Lemma:

Lemma 3 Let (55) hold for $t \in [0, T]$ and ϵ be such that

$$\epsilon \sup_{t \in [0, T]} \|w(t)\|_{L_\varphi^2} < 1. \quad (56)$$

$$1 - C\epsilon^{\frac{1}{2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \geq 0 \quad (57)$$

Then,

$$\|w(t)\|_{L_\varphi^2}^2 \leq C\epsilon^{-1} \left(k \left(|c|^{\frac{3}{2}} + |c| \right) + C\|\tilde{g}\|_{L_b^2} + \|w(0)\|_{L_b^2} \right)^2 \quad (58)$$

Proof:

We argue analogously as in Anthony and Zelik, 2014 to prove the validity of (56), therefore, We drop out the second term on the LHS of (55) to obtain:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|w\|_{L_\varphi^2}^2 - (w, v_\varphi) \right) + \gamma \left(\frac{1}{2} \|w\|_{L_\varphi^2}^2 - (w, v_\varphi) \right) \leq \frac{C}{\epsilon} k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + k(|c|^{\frac{5}{2}} + |c|)^2 + \\ & C\|\tilde{g}\|_{L_\varphi^2}^2 + \sup_t \|w(t)\|_{L_\varphi^2}^2. \end{aligned} \quad (59)$$

By Gronwall inequality, we have:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|w\|_{L_\varphi^2}^2 - (w, v_\varphi) \right) \leq \left(\frac{1}{2} \|w_0\|_{L_\varphi^2}^2 - (w_0, v_{0\varphi}) \right) e^{-\gamma t} + \frac{1}{\gamma} \left(\frac{C}{\epsilon} k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + k \left(|c|^{\frac{5}{2}} + |c| \right)^2 + \right. \\ & \left. C\|\tilde{g}\|_{L_\varphi^2}^2 \right) + \int_0^T e^{-\gamma(s-t)} \sup_{t \in [0, T]} \|w(s)\|_{L_\varphi^2}^2 ds + C. \end{aligned} \quad (60)$$

By taking supremum and doing Cauchy-Schwartz on (w, v_φ) and $(w_0, v_{0\varphi})$, we obtain the following:

$$\sup_{t \in [0, T]} \frac{1}{2} \|w(t)\|_{L^2_\varphi}^2 \leq \frac{1}{2} \|w_0\|_{L^2_\varphi}^2 + \sup_{t \in [0, T]} \|w(t)\|_{L^2_\varphi}^2 + C + \frac{1}{\gamma} \left(\frac{C}{\epsilon} k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + k \left(|c|^{\frac{5}{2}} + |c| \right)^2 + C \|\tilde{g}\|_{L^2_\varphi}^2 \right) + 2\epsilon \sup_{t \in [0, T]} \|w(t)\|_{L^2_\varphi}^2. \quad (61)$$

$$\sup_{t \in [0, T]} \frac{1}{2} \|w(t)\|_{L^2_\varphi}^2 \leq \epsilon^{-1} \left(\frac{C}{\epsilon} k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + \epsilon k \left(|c|^{\frac{5}{2}} + |c| \right)^2 + C \|\tilde{g}\|_{L^2_\varphi}^2 + \|w(0)\|_{L^2_b}^2 + \epsilon C \right) \quad (62)$$

$$\sup_{t \in [0, T]} \frac{1}{2} \|w(t)\|_{L^2_\varphi}^2 \leq C\epsilon^{-1} \left(\frac{C}{\epsilon} k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + C \|\tilde{g}\|_{L^2_b}^2 + \|w(0)\|_{L^2_b}^2 + 1 \right). \quad (63)$$

We conclude that:

$$\sup_{t \in [0, T]} \frac{1}{2} \|w(t)\|_{L^2_\varphi}^2 \leq C\epsilon^{-1} (k \left(|c|^{\frac{3}{2}} + |c| \right) + \|\tilde{g}\|_{L^2_b} + \|w(0)\|_{L^2_b}^2 + 1)^2. \quad (64)$$

This proves **Lemma 3**.

Let ϵ be such that estimate (63) satisfy:

$$\epsilon \sup_{t \in [0, T]} \|w(t)\|_{L^2_\varphi} < \frac{1}{2} \quad (65)$$

and,

$$C\epsilon^{\frac{1}{2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \leq \frac{1}{2}; \quad (66)$$

therefore, (65)-(66) is justified if ϵ satisfies the following:

$$\epsilon \sim \frac{\frac{1}{2}}{(k \left(|c|^{\frac{3}{2}} + |c| \right) + \|\tilde{g}\|_{L^2_b} + \|w(0)\|_{L^2_b}^2)^2} \leq \frac{\frac{1}{2}}{k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + 1}. \quad (67)$$

By analogous loop argument in Anthony, 2016 we prove (56) and by substituting (67) into (63) we obtain the final estimates in uniformly local spaces thus:

$$\sup_{t \in [0, T]} \|w(t)\|_{L^2_\varphi} \leq C(\|w_0\|_{L^2_b} + \|\tilde{g}\|_{L^2_b} + k \left(|c|^{\frac{3}{2}} + |c| \right) + 1)^4. \quad (68)$$

Now, integrating

$$\frac{d}{dt} \left(\frac{1}{2} \|w\|_{L^2_\varphi}^2 - (w, v_\varphi) \right) + C \|\nabla w\|_{L^2_\varphi}^2 \left(1 - \epsilon \sup_{t \in [0, T]} \|w(t)\|_{L^2_\varphi} \right) + \gamma \|w\|_{L^2_\varphi}^2 \leq \frac{C}{\epsilon} k \left(|c|^{\frac{3}{2}} + |c| \right)^2 + k \left(|c|^{\frac{5}{2}} + |c| \right)^2 + C \|\tilde{g}\|_{L^2_\varphi}^2 + C\epsilon^{\frac{1}{2}} k \left(|c|^{\frac{3}{2}} + |c| \right) \sup_t \|w(t)\|_{L^2_\varphi}^2.$$

in time and using

$$\sup_{t \in [0, T]} \frac{1}{2} \|w(t)\|_{L^2_\varphi}^2 \leq C\epsilon^{-1} (k \left(|c|^{\frac{3}{2}} + |c| \right) + \|\tilde{g}\|_{L^2_b} + \|w(0)\|_{L^2_b}^2 + 1)^2$$

we obtain the following estimate:

$$\|w\|_{L^\infty(0, T; L^2_b)} + \|\nabla w\|_{L^2_b(0, T; L^2_b)} \leq C \left(\|w_0\|_{L^2_b} + \|\tilde{g}\|_{L^2_b} + k \left(|c|^{\frac{3}{2}} + |c| \right) + 1 \right)^2.$$

Conclusion

A sufficiently regular solution of Navier-Stokes equation with nonzero flux condition in infinite strip, together with its derivative, are bounded in L_b^2 and L^∞ for all time.

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